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Correlations and Spectra for Non-Stationary Random Functions

By J. Kampé de Fériet and François N. Frenkiel

1. Introduction. The definition of the correlation function and the spectrum of a stationary random function is now classical, but in many applications one feels the need to extend this definition to random functions, which, although non-stationary, are in some sense *nearly* stationary. We suggest, therefore, for the definition of the correlation of a random function whose covariance $\Gamma(t, s)$ is known, the limit

$$R(h) = \lim_{T \to +\infty} \frac{1}{T} \int_{\frac{|h|}{2}}^{T-\frac{|h|}{2}} \Gamma\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) d\xi, \tag{1.1}$$

if this limit exists for every h. The spectrum $S(\lambda)$ can then be obtained from $R(h)$ in the classical way.

We are led to the above definition of the *correlation function* $R(h)$ by the following considerations: we determine the *sample-correlation* from a truncated sample of the random function; we then obtain a *sub-correlation*, $R_T(h)$, of the random function (defined as the correlation of the truncated random function) by averaging the sample correlations; finally, the correlation $R(h)$ is defined by (1.1) as the limit of $R_T(h)$, if this limit exists.

The function $R(h)$, so defined, has all the properties of a correlation function. If the random function is stationary (wide sense) [4, p. 95–96], our definition coincides with the classical definition. The estimation of the correlation of a stationary random function has been considered extensively in the literature, particularly by U. Grenander and M. Rosenblatt [6], R. B. Blackman and J. W. Tukey [1], and E. Parzen [13, 14]. In order to evaluate how good the estimate $R(h)$ is from the sample-correlations $\rho_T(h, \omega)$, which are the only experimental observables, we compute the variance of the random variables $\rho_T(h, \omega)$ about $R_T(h)$, and then we compute (for a fixed h) an upper bound of $R(h) - R_T(h)$ for large T.

This paper is especially concerned with the case in which the random function has a periodic covariance $\Gamma(t + \tau, s + \tau) = \Gamma(t, s)$. To appreciate the scope of the above condition, let us note that it is always satisfied when the random function is a sum of two uncorrelated random functions, one being a stationary (wide sense) random function and the other a periodic random function.

The last part of the paper is devoted to the estimate of $R(h)$ for a non-stationary random step-function $V(t, \omega)$, similar to the one introduced by N. Wiener,

$$V(t, \omega) = X_n(\omega), \quad n - 1 \leqq t < n, n = 1, 2, \cdots, \tag{1.2}$$

where $[X_1(\omega), \cdots X_n(\omega), \cdots]$ is a sequence of independent random variables taking only the values -1 and $+1$ with equal probability.

An experimental function will be constructed using a table of random numbers, and sample-correlations will be determined. Estimates for the sub-correlations are

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then determined by taking averages over the experimental correlations. The accuracy of these estimates is characterized by giving the variance for the departure of the sample-correlations from the estimated sub-correlations. The experimental data are then compared with the theoretical results. (Cf. [5], [9].)

2. Random Functions, Covariances and Correlation Functions. We consider here a *random function* of a real variable t as an *ensemble* of real functions $f(t, \omega)$, where ω is a parameter chosen at random in some set Ω according to a probability measure μ [7]. A *sample* of the random function $f(t, \omega)$ is simply the real function $f(t, \omega_0)$ corresponding to a particular choice of ω_0 in the set Ω. It is convenient for many applications to take for Ω a *function space*; each point ω is then a function $\omega(t)$ belonging to some prescribed class of functions (e.g., a continuous function on $[0, 1]$). One has thus for each sample $f(t, \omega) = \omega(t)$. When this particular choice is made for Ω, one says that the random function is of "function space type" [4, p. 67].

The following general hypotheses shall be made with regard to the random functions considered in the present paper:

$\mathbf{H}_1$. $f(t, \omega)$ is measurable with respect to the product measure $m \times \mu$ (where m is the Lebesgue measure on the real line $-\infty < t < +\infty$).

$\mathbf{H}_2$. For each t, $f(t, \omega) \in L^2(\Omega)$: $\overline{f(t, \omega)^2} < +\infty$. (If $F(\omega) \in L(\Omega)$ we denote its mean value by $\int_\Omega F(\omega) d\mu = \overline{F(\omega)}$.)

$\mathbf{H}_3$. $\mathbf{H}_2$ implies $f(t, \omega) \in L(\Omega)$; we suppose $\overline{f(t, \omega)} = 0$.

$\mathbf{H}_4$. It follows from $\mathbf{H}_2$ that the *covariance*

$$\Gamma(t, s) = \overline{f(t, \omega) f(s, \omega)} \tag{2.1}$$

exists for all t's and s's. We assume that,

$$\Gamma(t, t) \in L[a, b] \tag{2.2}$$

for every finite interval $a \leqq t \leqq b$. From (2.2), by the Fubini-Tonelli theorem, [8, Vol. 1, p. 609] it follows that

$$f(t, \omega) \in L^2[a, b] \tag{2.3}$$

for almost all samples, in any finite interval $a \leqq t \leqq b$. This implies also that

$$f(t, \omega) \in L[a, b] \qquad \text{for almost all samples.} \tag{2.4}$$

In our earlier paper [9] instead of (2.2), we assumed that

$$F(t, s) \in L(\Delta^2) \tag{2.2'}$$

for every finite rectangle Δ^2 in the plane (t, s). We are indebted to the referee for a simple counter example showing that (2.2') does not always imply (2.3) and for the suggestion that we replace (2.2') by (2.2); the proof that (2.2) implies (2.3) by the Fubini-Tonelli theorem is straightforward.

If the random function $f(t, \omega)$ is stationary

$$\Gamma(t, s) = \rho(t - s) \tag{2.5}$$

where $\rho(h)$ is called a *correlation function*. Due to $\mathbf{H}_1$, $\rho(h)$ is uniformly continuous in any finite interval [3]. A real function $\rho(h)$ is a correlation function if, and only

if, it is symmetric and positive-definite

$$\rho(-h) = \rho(h) \tag{2.6}$$

$$\sum_j \sum_k X_j X_k \rho(h_j - h_k) \geqq 0 \tag{2.7}$$

for any set $[h_1, \cdots h_n]$ with n arbitrary.

3. Truncated Samples of Random Functions. In experiments concerning a random function $f(t, \omega)$ one can materialize, as a rule, the sample of the function only for a finite interval, that is, one knows only *truncated samples*. As far as finite intervals are considered, one often uses the notation $[-T, +T]$ for the interval in which the samples are known in the experiment. Rather than this *two-sided* (symmetric with respect to $t = 0$) *truncation*, we shall prefer here a *one-sided truncation* (starting at $t = 0$) and we will define a *truncated sample* by

$$\begin{aligned} f_T(t, \omega_0) &= f(t, \omega_0), & 0 \leqq t \leqq T, \\ f_T(t, \omega_0) &= 0, & t < 0 \text{ or } t > T. \end{aligned} \tag{3.1}$$

This definition implies that the experiment starts at $t = 0$; we assume that it could be extended for an arbitrary time T in the future, but not in the past (time prior to the beginning of the experiment). From the samples we will draw some inference with regard to the random function for $0 \leqq t < +\infty$, but completely ignore it for $t < 0$.

4. Correlation and Spectrum of a Truncated Sample. For a truncated sample, corresponding to a given ω_0, we define a *sample correlation* as

$$\rho_T(h, \omega_0) = \frac{1}{T} \int_{|h|/2}^{T-|h|/2} f\left(\xi - \frac{h}{2}, \omega_0\right) f\left(\xi + \frac{h}{2}, \omega_0\right) d\xi, \qquad \text{for} \quad |h| \leqq T, \tag{4.1}$$

and

$$\rho_T(h, \omega_0) = 0, \qquad \text{for } |h| \geqq T. \tag{4.2}$$

Let us remark that both (4.1) and (4.2) can be replaced by the formula

$$\begin{aligned} \rho_T(h, \omega_0) &= \frac{1}{T} \int_{-\infty}^{+\infty} f_T\left(\xi - \frac{h}{2}, \omega_0\right) f_T\left(\xi + \frac{h}{2}, \omega_0\right) d\xi \\ &= \frac{1}{T} \int_{-\infty}^{+\infty} f_T(\xi, \omega_0) f_T(\xi + |h|, \omega_0)\, d\xi, \qquad \text{for all } h. \end{aligned} \tag{4.3}$$

From $\mathbf{H}_2$ it follows that the correlation $\rho_T(h, \omega)$ exists for almost all samples (i.e., with probability one). The great advantage of our definition is that *the correlation $\rho_T(h, \omega)$ is a positive-definite function of h, uniformly continuous in h (for each ω for which it exists)*.

If we had used as correlation of the truncated sample, as is very often done, the function

$$\tilde{\rho}_T(h, \omega) = \frac{\rho_T(h, \omega)}{1 - \dfrac{|h|}{T}} \tag{4.4}$$

then we would have completely missed these important properties of $\rho_T(h, \omega)$. Indeed, $\tilde{\rho}_T(h, \omega)$ would have been, in general, discontinuous at $|h| = T$ and no longer a positive-definite function. Thus $\tilde{\rho}_T(h, \omega)$ would not have been the Fourier transform of a function $\psi_T(\lambda, \omega)$.

The only advantage gained by using $\tilde{\rho}_T(h, \omega)$ is that if f is constant, $\tilde{\rho}_T$ is also a constant while ρ_T is not. This fact was probably the reason which led the statisticians to use this definition for the correlation of a truncated function. However, the nonexistence of a spectrum $\psi_T(\lambda, \omega)$ which may have to be reintroduced later by various artifices, may lead to serious complications in the estimation of the correlation functions, particularly when numerical methods are used for that purpose.

The spectrum $\psi_T(\lambda, \omega)$ is very simply connected to the complex Fourier transform of the sample

$$\alpha_T(\lambda, \omega) = \frac{1}{T}\int_0^T e^{-i\lambda t} f(t, \omega)\, dt = \frac{1}{T}\int_{-\infty}^{+\infty} e^{-i\lambda t} f_T(t, \omega)\, dt. \tag{4.5}$$

Due to (2.3), the Fourier transform exists for almost all samples and, by Plancherel's theorem, [8, Vol. 2, p. 742], $\alpha_T(\lambda, \omega) \in L^2[-\infty, +\infty]$ (but not, in general, to $L[-\infty, +\infty]$). From (4.5) we have:

$$|\alpha_T(\lambda, \omega)|^2 = \frac{1}{T^2}\iint_{-\infty}^{+\infty} e^{i\lambda(s-t)} f_T(t, \omega) f_T(s, \omega)\, dt\, ds. \tag{4.6}$$

Let us consider the (t, s) plane and make the change of variables

$$t = \xi - \frac{h}{2}, \qquad s = \xi + \frac{h}{2}, \tag{4.7}$$

$$\xi = \frac{t+s}{2}, \qquad h = s - t.$$

For any $F(t, s) \in L(R^2)$

$$\iint_{-\infty}^{+\infty} F(t, s)\, dt\, ds = \iint_{-\infty}^{+\infty} F\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) d\xi\, dh \tag{4.8}$$

$$= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} F\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) d\xi\right] dh$$

the last formula being a consequence of Fubini's theorem [8, Vol. 1, p. 631].

Using this transformation, we can write (4.6) in the new form

$$|\alpha_T(\lambda, \omega)|^2 = \frac{1}{T}\int_{-\infty}^{+\infty} e^{i\lambda h} \rho_T(h, \omega)\, dh. \tag{4.10}$$

Using the fact that $\rho_T(-h, \omega) = \rho_T(h, \omega)$ we have

$$|\alpha_T(\lambda, \omega)|^2 = \frac{2}{T}\int_0^T \rho_T(h, \omega) \cos \lambda h\, dh. \tag{4.11}$$

Thus we obtain immediately

$$\psi_T(\lambda, \omega) = \frac{2}{\pi}\int_0^T \rho_T(h, \omega) \cos \lambda h\, db = \frac{T}{\pi}|\alpha_T(\lambda, \omega)|^2 \tag{4.12}$$

which gives the expression of the spectrum in terms of the Fourier transform of the truncated sample.

Let us note the following properties of this function

$$\psi_T(\lambda, \omega) \geqq 0, \tag{4.13}$$

$$\psi_T(-\lambda, \omega) = \psi_T(\lambda, \omega), \tag{4.14}$$

$$\psi_T(\lambda, \omega) \in L(0, +\infty). \tag{4.15}$$

Due to this last property we can invert the Fourier transform (4.12) and we obtain the reciprocal formula

$$\rho_T(h, \omega) = \int_{\mathbb{R}}^{+\infty} \psi_T(\lambda, \omega) \cos \lambda h \, d\lambda. \tag{4.16}$$

5. Sub-Correlation and Sub-Spectrum of a Random Function. Let us now define the *sub-correlation* $R_T(h)$ of the random function $f_T(t, \omega)$ as the average of the sample-correlations $\rho_T(h, \omega)$, i.e.,

$$R_T(h) = \overline{\rho_T(h, \omega)}. \tag{5.1}$$

Obviously, we have

$$R_T(h) = 0, \qquad |h| > T. \tag{5.2}$$

and for all other values of h

$$R_T(h) = \int_\Omega \left[\frac{1}{T} \int_{|h|/2}^{T-|h|/2} f\left(\xi - \frac{h}{2}, \omega\right) f\left(\xi + \frac{h}{2}, \omega\right) d\xi \right] d\mu, \qquad |h| \leqq T. \tag{5.3}$$

Inverting the double integral in accordance with Fubini's theorem we find

$$R_T(h) = \frac{1}{T} \int_{|h|/2}^{T-|h|/2} \Gamma\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) d\xi, \qquad 0 \leqq h \leqq T. \tag{5.4}$$

Let us refer to the change of variables (4.7) and let us consider the (t, s) plane (Figure 1). We introduce the following notation

$$\Delta_T = \{(t, s): 0 \leqq t \leqq T, \quad 0 \leqq s \leqq T\}. \tag{5.5}$$

Then obviously

$$R_T(h) = \frac{1}{T} \times \text{Integral of } \Gamma(t, s) \text{ along the segment } \overline{AB} \text{ of } \delta(h) \text{ contained in } \Delta_T. \tag{5.6}$$

If we define

$$\begin{aligned} \Gamma_T(t, s) &= \Gamma(t, s), & (t, s) &\in \Delta_T \\ \Gamma_T(t, s) &= 0, & (t, s) &\notin \Delta_T \end{aligned} \tag{5.7}$$

we can also write (5.4) in the following way

$$R_T(h) = \frac{1}{T} \int_{-\infty}^{+\infty} \Gamma_T\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) d\xi, \tag{5.8}$$

which is true for all values of h, giving $R_T(h) = 0$ for $h < T$ or $h > T$. Obviously,

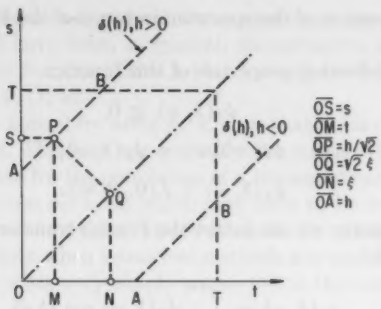

FIG. 1.—Change of variables.

one has also

$$R_T(h) = \frac{1}{T}\int_{-\infty}^{+\infty} \Gamma_T(\xi, \xi + |h|)\, d\xi$$

$$= \frac{1}{T}\int_{0}^{T-|h|} \Gamma(\xi, \xi + |h|)\, d\xi. \tag{5.9}$$

Thus, *to compute $R_T(h)$ for all h it is sufficient to know the covariance $\Gamma(t, s)$ in the square Δ_T.*

Let us define the *sub-spectrum of the random function* $f(t, \omega)$ as the average over the spectra of the truncated random function

$$\varphi_T(\lambda) = \overline{\psi_T(\lambda, \omega)}. \tag{5.10}$$

Then we have the two reciprocal formulas

$$R_T(h) = \int_0^{+\infty} \varphi_T(\lambda) \cos \lambda h\, d\lambda \tag{5.11}$$

$$\varphi_T(\lambda) = \frac{2}{\pi}\int_0^T R_T(h) \cos \lambda h\, dh. \tag{5.12}$$

From (4.13) it follows that $\varphi_T(\lambda) \geqq 0$; *thus, by S. Bochner's theorem* [2] $R_T(h)$ *is a continuous correlation function.*

6. Correlation Function of a Random Function. We define the *correlation function* $R(h)$ of the random function $f(t, \omega)$ by

$$R(h) = \lim_{T\to+\infty} R_T(h) = \lim_{T\to+\infty} \frac{1}{T}\int_{|h|/2}^{T-|h|/2} \Gamma\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) d\xi, \tag{6.1}$$

if this limit exists for every real h.

If the random function is stationary (wide sense), then by definition

$$\Gamma(t, s) = \rho(s - t), \qquad \Gamma\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) = \rho(h), \tag{6.2}$$

and

$$R_T(h) = \left(1 - \frac{|h|}{T}\right)\rho(h), \qquad |h| \leqq T, \tag{6.3}$$

$$R_T(h) = 0, \qquad |h| \geqq T.$$

Thus the limit of $R_T(h)$ exists and

$$R(h) = \lim_{T\to\infty}\left(1 - \frac{|h|}{T}\right)\rho(h) = \rho(h). \tag{6.4}$$

Hence, for a stationary random function our definition gives the classical result, but we can also apply (6.1) to non-stationary random functions.

Let us consider as an example the random function

$$f(t, \omega) = \frac{W(t)}{\sqrt{t}}$$

where $W(t)$ is the classical Wiener-Lévy function, giving the abscissa at time t of a particle, starting from the origin at time $t = 0$, and subjected to one-dimensional Brownian motion. This function is certainly not stationary; it has the covariance

$$\Gamma(t, s) = \sqrt{\frac{t}{s}}, \qquad 0 < t \leqq s.$$

$$\Gamma(t, s) = \sqrt{\frac{s}{t}}, \qquad 0 < s \leqq t.$$

According to our definition this nonstationary random function has the correlation

$$R(h) = \lim_{T\to+\infty}\frac{1}{T}\int_0^{T-|h|}\sqrt{\frac{\xi}{\xi + |h|}}\,d\xi = 1, \qquad \text{for all } h.$$

7. Spectrum of a Non-Stationary Random Function. As far as the spectrum is concerned, $\varphi_T(\lambda)$ does not, in general, tend toward a limit when $T \to +\infty$, even if the correlation $R(h)$ exists, but exactly as in the stationary case [10, Vol. 2, p. 164–166] it can be shown, using Paul Lévy's continuity theorem [12, p. 195], that the integrated spectrum

$$S_T(\lambda) = \int_0^\lambda \varphi_T(\eta)\,d\eta \tag{7.1}$$

does in fact tend toward a limit

$$S(\lambda) = \lim_{T\to+\infty} S_T(\lambda) \tag{7.2}$$

if the correlation $R(h)$ defined by (6.1) exists and is continuous. (This is not necessarily true; $R(h)$ being the limit of a sequence of continuous functions can be discontinuous.) Thus,

$$R(h) = \int_0^{+\infty} \cos \lambda h\, dS(\lambda), \tag{7.3}$$

this being a Fourier-Stieltjes integral and the spectrum $S(\lambda)$ being a monotonic

non-decreasing function such that $S(0) = 0$, $S(+\infty) < +\infty$. In general, $S(\lambda)$ is discontinuous and the spectrum has a countable number of *lines* corresponding to a finite amount of energy.

The spectrum $S(\lambda)$ can be computed from the correlation $R(h)$ by applying the Paul Lévy inversion formula [12, p. 166]

$$S(\lambda_2) - S(\lambda_1) = \lim_{A \to +\infty} \frac{2}{\pi} \int_0^A \frac{\sin \lambda_2 h - \sin \lambda_1 h}{h} R(h)\, dh, \tag{7.4}$$

which is valid if λ_1 and λ_2 are continuity points of $S(\lambda)$. When $R(h)$ is known the spectrum is thus defined for every $\lambda > 0$ with the exception of, at most, a countable number of discontinuity points.

8. Estimation of Correlations for a Non-Stationary Function. For a non-stationary random function, even if we have not only truncated samples of the function, but also its covariance in Δ_T, this does not give us sufficient information to determine $R(h)$. It is obvious, from (6.1), that large values of ξ are most important in determining $R(h)$ (even at small values of h). The knowledge of $\Gamma(t, s)$ in the square Δ_T only does not give us any information about its values for large ξ on $\delta(h)$ (See Fig. 1).

We shall consider here one class of random functions which is not stationary, but on which information is given, which enables us to make an estimate of $R(h)$.

This class is defined by the condition that $\Gamma\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right)$ *is periodic in* ξ. This condition means that the covariance is invariant under a translation τ parallel to the first bissectrix (Fig. 1)

$$\Gamma(t + \tau, s + \tau) = \Gamma(t, s). \tag{8.1}$$

The scope of the implications of this hypothesis for applications is better understood if one points out that (8.1) is satisfied when the phenomenon represented by the function $f(t, \omega)$ is the result of the superposition of two phenomena, one stationary and the other periodic. Thus

$$f(t, \omega) = f_1(t, \omega) + f_2(t, \omega) \tag{8.2}$$

where $f_1(t, \omega)$ is a stationary (wide sense) random function:

$$\overline{f_1(t, \omega) f_1(s, \omega)} = \rho_1(t - s) \tag{8.3}$$

and $f_2(t, \omega)$ is periodic

$$f_2(t + \tau, \omega) = f_2(t, \omega) \tag{8.4}$$

$$\begin{aligned} \overline{f_2(s, \omega) f_2(t, \omega)} &= \Gamma_2(t, s) \\ \Gamma_2(t + \tau, s) &= \Gamma_2(t, s + \tau) = \Gamma_2(t, s), \end{aligned} \tag{8.5}$$

the two random functions f_1 and f_2 being uncorrelated

$$\overline{f_1(t, \omega) f_2(s, \omega)} = 0. \tag{8.6}$$

Thus one has

$$\Gamma(t, s) = \rho_1(t - s) + \Gamma_2(t, s) \tag{8.7}$$

and $\Gamma(t, s)$ satisfies (8.1).

In this case the limit (6.1) obviously exists and the correlation is given by

$$R(h) = \frac{1}{\tau}\int_0^\tau \Gamma(\xi, \xi + |h|)\, d\xi. \tag{8.8}$$

The sub-correlation for $T = N\tau$ (where N is an integer), due to (5.9), is equal to

$$R_{N\tau}(h) = \frac{1}{N\tau}\int_0^{N\tau-|h|} \Gamma(\xi, \xi + |h|)\, d\xi. \tag{8.9}$$

As a consequence of the periodicity

$$R(h) - R_{N\tau}(h) = \frac{1}{N\tau}\int_0^{|h|} \Gamma(\xi, \xi + |h|)\, d\xi.$$

We thus have the upper bound

$$|R(h) - R_{N\tau}(h)| \leqq \frac{1}{N\tau}\int_0^{|h|} |\Gamma(\xi, \xi + |h|)|\, d\xi \tag{8.10}$$

for the error obtained by using the sub-correlation $R_{N\tau}(h)$ instead of the correlation $R(h)$. The upper bound (8.10) is a function of h; however, at a given h, this bound tends to 0 as $1/N$.

Let us consider the case when $(k-1)\tau \leqq |h| \leqq k\tau$ (k integer). Using the periodicity of Γ and Schwarz inequality we have

$$|R(h) - R_{N\tau}(h)| \leqq \frac{k}{N}\frac{1}{\tau}\int_0^\tau \Gamma(\xi, \xi)\, d\xi. \tag{8.11}$$

When the covariance $\Gamma(t, s)$ is known for one period over the diagonal $t = s$ we can compute

$$A = \frac{1}{\tau}\int_0^\tau \Gamma(\xi, \xi)\, d\xi. \tag{8.12}$$

Finally, the upper bound for the approximation is given by

$$|R(h) - R_{N\tau}(h)| \leqq A\frac{k}{N}, \qquad (k-1)\tau \leqq h \leqq k\tau. \tag{8.13}$$

Let us now consider $R_T(h)$ when $N\tau < T < (N+1)\tau$. We have

$$\begin{aligned} R_T(h) - \frac{N\tau}{T}R_{N\tau}(h) &= \frac{1}{T}\int_{N\tau-|h|}^{T-|h|} \Gamma(\xi, \xi + |h|)\, d\xi \\ &= \frac{1}{T}\int_{N\tau}^{T} \Gamma(\xi, \xi + |h|)\, d\xi. \end{aligned}$$

Thus, due to the periodicity of Γ,

$$\left| R_T(h) - \frac{N\tau}{T}R_{N\tau}(h) \right| \leqq A\frac{\tau}{T} < \frac{A}{N}. \tag{8.14}$$

From (8.13) and (8.14) we obtain finally

$$\left| R_T(h) - \frac{N_T}{T} R(h) \right| \leqq A \frac{1+k}{N}, \qquad N_T < T < (N+1)_T, \tag{8.15}$$

$$(k-1)_T \leqq h < k_T.$$

This relation shows that *the approximation is good if k/N is small.*

9. Accuracy of Estimates of Correlations. In estimating the correlation $R(h)$ of a random function from truncated samples there are two steps:

(a) From the correlations $\rho_T(h, \omega_j)$ of the truncated samples we estimate the sub-correlation $R_T(h)$,

(b) From $R_T(h)$ we compute $R(h)$.

In the preceding section for the case of a random function with a periodic covariance we have solved problem (b) at large values of T. Now, let us look at the problem (a), namely, how to evaluate the approximation with which one determines $R_T(h)$ from the average of the correlations $\rho_T(h, \omega_j)$ of a number q of samples

$$\hat{R}_{T,q}(h) = \frac{1}{q} \sum_{j=1}^{j=q} \rho_T(h, \omega_j). \tag{9.1}$$

It appears that the best way to make such an evaluation is to determine the variance of the random variables $\rho_T(h, \omega_j)$ about their mean value $R_T(h)$; if this variance is small enough we can expect that for a reasonably large number q of samples the estimate $\hat{R}_T(h)$ will be fairly good.

In order to compute this variance, in addition to Hypotheses $\mathbf{H}_1$ to $\mathbf{H}_4$ of Section 2, we shall assume that

$\mathbf{H}_5$. $f(t, \omega) \in L^4(\Omega)$ for all t's.

This insures the existence of the fourth-order moment

$$\mathfrak{M}(t_1, t_2, t_3, t_4) = \int_\Omega f(t_1, \omega) f(t_2, \omega) f(t_3, \omega) f(t_4, \omega)\, d\mu \tag{9.2}$$

for all $[t_1, t_2, t_3, t_4]$.

We shall, moreover, assume that

$\mathbf{H}_6$. $\mathfrak{M}(t_1, t_2, t_3, t_4) \in L(\Delta)$ for every finite parallelepiped of the four-dimensional space (t_1, t_2, t_3, t_4).

Let us observe that the fourth-order moment exists in the important case of normal random functions, i.e., when the n random variables, $f(t_1, \omega), \cdots f(t_n, \omega)$ $[t_1, \cdots, t_n]$ arbitrary follow an n-variate normal (Gaussian) law.

From (4.3) and (5.8) we find that the departure of a correlation for a particular truncated sample from the mean value taken over the correlations for all samples is given by

$$\rho_T(h, \omega) - R_T(h) = \frac{1}{T} \int_{-\infty}^{+\infty} \left[\Gamma_T\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right) - f_T\left(\xi - \frac{h}{2}, \omega\right) f_T\left(\xi + \frac{h}{2}, \omega\right) \right] d\xi.$$

After taking a square and averaging, we obtain

$$\sigma_T(h)^2 = \overline{[\rho_T(h,\omega) - R_T(h)]^2} = \frac{1}{T^2}\iint_{-\infty}^{+\infty} F_T(\xi, \eta, h)\, d\xi\, d\eta \tag{9.3}$$

with

$$F_T(\xi, \eta, h) = \mathfrak{M}_T\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}, \eta - \frac{h}{2}, \eta + \frac{h}{2}\right) - \Gamma_T\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right)\Gamma_T\left(\eta - \frac{h}{2}, \eta + \frac{h}{2}\right). \tag{9.4}$$

In the above equation $\Gamma_T(t, s)$ is defined by (5.7) and $\mathfrak{M}_T$ by the two relations

$$\mathfrak{M}_T(t_1, t_2, t_3, t_4) = \mathfrak{M}(t_1, t_2, t_3, t_4) \quad \text{when} \quad (t_1, t_2) \in \Delta_T \quad \text{and} \quad (t_3, t_4) \in \Delta_T \tag{9.5}$$

and

$$\mathfrak{M}_T(t_1, t_2, t_3, t_4) = 0 \quad \text{when} \quad (t_1, t_2) \notin \Delta_T \quad \text{or} \quad (t_3, t_4) \notin \Delta_T. \tag{9.6}$$

Equation (9.4) shows that whenever the covariance Γ and the fourth-order moment $\mathfrak{M}$ are known, we can evaluate the variance, $\sigma_T(h)^2$, of the random variables $\rho_T(h, \omega)$ about their mean value $R_T(h)$. In particular, for $h = 0$, we have

$$\sigma_T(0)^2 = \frac{1}{T^2}\iint_{\Delta_T} [\mathfrak{M}(\xi, \xi, \eta, \eta,) - \Gamma(\xi, \xi)\Gamma(\eta, \eta)]\, d\xi\, d\eta. \tag{9.7}$$

10. Random Step- Function with Periodic Covariance. As an example let us take the random function defined as follows*

$$V(t, \omega) = X_n(\omega) \quad n - 1 \leqq t < n, n = 1, 2, \cdots \tag{10.1}$$

where

$$X_1(\omega), \cdots, X_n(\omega), \cdots \tag{10.2}$$

is a sequence of independent random variables, taking only the values -1 and $+1$ with equal probability

$$\text{Prob}\,[X_n = -1] = \text{Prob}\,[X_n = +1] = \tfrac{1}{2}. \tag{10.3}$$

The random function $V(t, \omega)$ is essentially the same as a function considered by Norbert Wiener in his pioneering work on correlation and spectrum, [15, p. 151] except for the fact that his function is defined for $-\infty < t < +\infty$ and ours only in $[0, +\infty]$.

From (10.3) it follows that

$$\bar{X}_n = 0 \tag{10.4}$$

$$\overline{X_m X_n} = \delta_{m,n}, \tag{10.5}$$

* As measure space Ω, one can take the interval $[0, 1]$, $0 \leqq \omega < 1$, with $X_n(\omega) = 2a_n - 1$, where a_n is the nth coefficient in the binary development $\omega = a_1/2 + \cdots + a_n/2^n + \cdots$. The measure μ on $[0, 1]$ is the Lebesgue measure $\mu\{\omega: a_n = 1\} = \mu\{\omega: a_n = 0\} = \frac{1}{2}$.

where $\delta_{m,n}$ is the Kronecker symbol. If one defines

$$\Delta_{m,n} = \{(t, s): m - 1 \leqq t < m, n - 1 \leqq s < n\}$$

then

$$\Gamma(t, s) = 1, \qquad (t, s) \in \bigcup_{n=1}^{n=+\infty} \Delta_{n,n},$$

$$\Gamma(t, s) = 0, \qquad (t, s) \notin \bigcup_{n=1}^{n=+\infty} \Delta_{n,n}.$$

Obviously, the random function $V(t, \omega)$ is not stationary, but its covariance, when put in the form $\Gamma\left(\xi - \frac{h}{2}, \xi + \frac{h}{2}\right)$ is periodic, with period 1 in ξ (see Fig. 2). Thus the results of Section 8 apply, and the non-stationary random function $V(t, \omega)$ has, according to (8.8), a correlation

$$\begin{aligned} R(h) &= (1 - |h|), && |h| \leqq 1, \\ R(h) &= 0, && |h| \geqq 1. \end{aligned} \tag{10.6}$$

Let us first consider the case $T = N$, where N is an integer. We find that

$$\begin{aligned} \rho_N(h, \omega) &= (k - |h|)Y_N(k - 1, \omega) + (1 - k + |h|)Y_N(k, \omega), \\ & \qquad k - 1 \leqq |h| \leqq k, \qquad 1 \leqq k \leqq N - 1, \\ \rho_N(h, \omega) &= 0, \qquad |h| \geqq N, \end{aligned} \tag{10.7}$$

where $Y_N(0, \omega), Y_N(1, \omega) \cdots Y_N(N - 1, \omega)$ represent the random variables

$$Y_N(k, \omega) = \frac{1}{N} \sum_{j=1}^{j=N-k} X_j(\omega) X_{j+k}(\omega), \quad k = 0, 1, \cdots (N - 1) \tag{10.8}$$

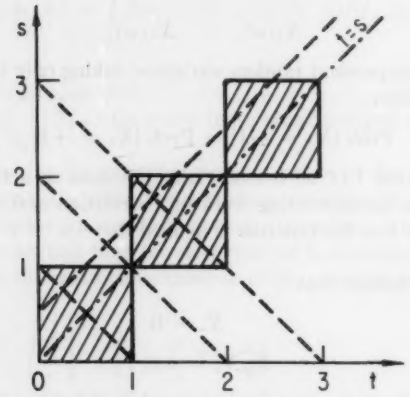

FIG. 2.—The (t, s) diagram for a random step-function. $\Gamma(\xi - \frac{h}{2}, \xi + \frac{h}{2})$ is periodic in ξ and for a translation parallel to $t = s, \Gamma(t + 1, s + 1) = \Gamma(t, s)$.

in particular

$$Y_N(0, \omega) = 1. \tag{10.9}$$

For each sample ω, the sample-correlation $\rho_N(h, \omega)$ is represented by a polygonal line (Fig. 3). The ordinates $Y_N(1, \omega)$, $Y_N(2, \omega) \cdots Y_N(N-1, \omega)$ corresponding to $h = 1, 2, \cdots (N-1)$, are the random variables defined by (10.8). We see immediately that $Y_N(k, \omega)$ can only take the values

$$-\frac{N-k}{N}, \quad -\frac{N-k-2}{N}, \quad \cdots +\frac{N-k-2}{N}, \quad +\frac{N-k}{N}$$

following the binomial law. As a consequence

$$|\rho_N(k, \omega)| \leqq 1 - \frac{k}{N}, \qquad k = 1, 2, \cdots, (N-1). \tag{10.10}$$

We obtain for the mean, variances, and covariances of the ordinates $Y_N(k, \omega)$ respectively,

$$\overline{Y_N(k, \omega)} = 0 \qquad k = 1, 2, \cdots, (N-1) \tag{10.11}$$

$$\overline{Y_N(k, \omega)^2} = \frac{N-k}{N^2}, \tag{10.12}$$

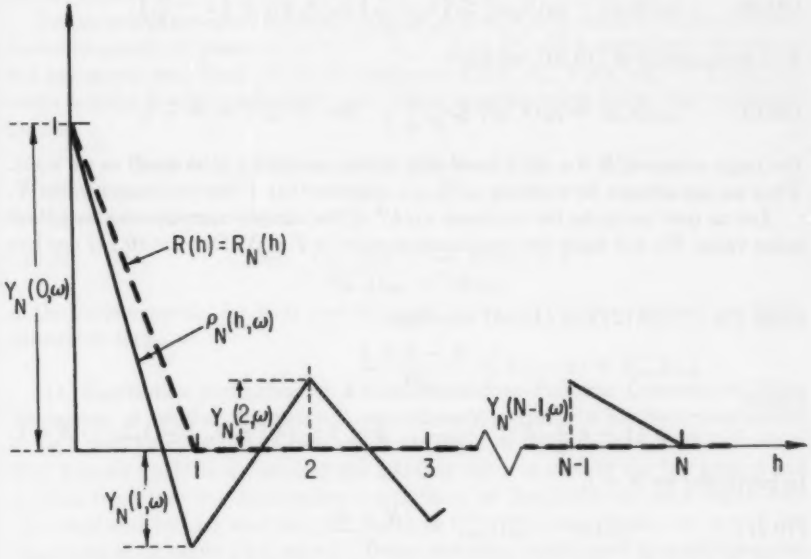

FIG. 3.—The polygonal line $\rho_N(h, \omega)$ represents a sample-correlation of the step-function. $R_N(h) = \overline{\rho_N(h, \omega)}$ is the sub-correlation, which for the step-function, is equal to the correlation function $R(h)$.

and

$$\overline{Y_N(k,\omega)Y_N(l,\omega)} = 0, \qquad k \neq l. \tag{10.13}$$

From (10.7), (10.9) and (10.11) we deduce

$$\begin{aligned} \overline{\rho_N(h,\omega)} &= 1 - |h|, & |h| \leqq 1 \\ \overline{\rho_N(h,\omega)} &= 0, & |h| \geqq 1. \end{aligned} \tag{10.14}$$

Thus

$$R_N(h) = \overline{\rho_N(h,\omega)} = R(h). \tag{10.15}$$

Let us next suppose $N \leqq T < N+1$. We find

$$\rho_T(h,\omega) = \frac{N}{T}\rho_N(h,\omega) + \frac{k-N+h}{T}X_k X_{N+1} + \frac{T-k-h}{T}X_{k+1}X_N, \tag{10.16}$$

$$N-k < h < T-k, \qquad k = 1, 2, \cdots N.$$

$$\rho_T(h,\omega) = \frac{N}{T}\rho_N(h,\omega) + \frac{T-N}{T}X_k X_{N+1}, \tag{10.17}$$

$$T-k < h < N+1-k, \qquad k = 1, 2, \cdots N.$$

Thus if $N < T < N+1$,

$$|\rho_T(h,\omega) - \rho_N(h,\omega)| \leqq \left(1 - \frac{N}{T}\right)|\rho_N(h,\omega)| + \left(1 - \frac{N}{T}\right). \tag{10.18}$$

As a consequence of (10.10), we have

$$|\rho_T(h,\omega) - \rho_N(h,\omega)| \leqq \frac{2}{N+1} \quad \text{for} \quad N \leqq T < N+1. \tag{10.19}$$

For large values of N the right-hand side of this inequality is as small as we want. Thus we can always, in studying $\rho_T(h,\omega)$, suppose that T has an integer value N.

Let us now compute the variance $\sigma_N(h)^2$ of the sample-correlations about their mean value. We will make the computation only for $T = N$. Due to (10.14) one has

$$\sigma_N(h)^2 = \overline{\rho_N(h,\omega)^2}.$$

From (10.7), (10.12) and (10.13) we obtain

$$\overline{\rho_N(h,\omega)^2} = (k - |h|)^2 \frac{N-k+1}{N^2} + (1 - k + |h|)^2 \frac{N-k}{N^2}, \tag{10.20}$$

$$k-1 \leqq |h| \leqq k, \quad k = 1,2, \cdots N-1.$$

In particular for $h = k$

$$\sigma_N(k)^2 = \overline{\rho_N(k,\omega)^2} = \frac{(N-k)}{N^2}. \tag{10.21}$$

The proposition that the *random variables* $\rho_N(h,\omega)$ *tend toward their limit* $R(h)$ *for almost all samples,*

$$\text{Prob}\left[\lim_{N\to+\infty} \rho_N(h,\omega) = R(h)\right] = 1,$$

will now be established. First from (10.8) we compute

$$\overline{Y_N(k,\omega)^4} = \frac{(N-k)(2N-2k+1)}{N^4}. \tag{10.21}$$

Let us now observe that for a fixed N, when k takes the values $1, 2, \cdots (N-1)$ one has $\overline{Y_N(k,\omega)^4} < \frac{3}{N^2}$. Thus, k now being fixed, we have

$$\sum_{N=1}^{N=+\infty} \overline{Y_N(k,\omega)^4} \leqq 3 \sum_{N=1}^{N=+\infty} \frac{1}{N^2} < +\infty$$

and, therefore

$$\text{Prob}\left[\sum_{N=1}^{N=+\infty} Y_N(k,\omega)^4 < +\infty\right] = 1.$$

Here, we use the following criterion for the almost sure absolute convergence for a series of random variables. If $\sum_1^{+\infty} |X_n| < +\infty$, then Prob $[\sum_1^{+\infty} |X_n| < +\infty] = 1$. This criterion applies even when the random variables $X_1, \cdots, X_2, \cdots$ are not independent. However, the convergence of the series implies that $Y_N(k,\omega) \to 0$. We have thus proved that

$$\text{Prob}\left[\lim_{N\to+\infty} Y_N(k,\omega) = 0\right] = 1$$

for each fixed $k \geqq 1$.

Let us now take a fixed interval $|h| \leqq M$, where M is an integer. In this interval, there are exactly M points, $h = 1, h = 2, \cdots h = M$, which completely determine the polygonal line. Each of the M ordinates $Y_N(1,\omega)$, $Y_N(2,\omega) \cdots Y_N(M,\omega)$ tends toward 0 with probability one. Their number being finite, this evidently implies

$$\text{Prob}\left[\lim_{N\to+\infty} Y_N(k,\omega) = 0 \quad \text{for} \quad k = 1, 2, \cdots M\right] = 1.$$

We have thus proved that

$$\text{Prob}\left[\lim_{N\to+\infty} \rho_N(h,\omega) = R(h)\right] = 1 \tag{10.22}$$

in the finite interval $|h| \leqq M$ and the proof is complete because M could be taken arbitrarily large.

11. Correlation Estimates for a Continuous Step-Function Constructed Using Sequences of Random Numbers. A continuous step-function has been constructed using a table of random numbers which are listed in 100 groups of 1000 digits each [11]. For our analysis we selected the 200 first digits of each of the 100 groups and divided them into five consecutive segments of 40 digits. All the even digits were then replaced by $+1$ and the odd digits by -1. We thus obtain one set of 500 sequences of 40 digits ($+1$, or -1). Other sets were constituted by combining two or more consecutive segments of 40 digits. As a result we have obtained the following five sets of experimental functions without overlapping sequences within each set: 500 sequences of 40 digits; 200 sequences of 80 digits; 100 sequences of 120, 160 and 200 digits, respectively.

Let us consider a set of q sequences with N digits in each sequence. A sample-correlation for the sequence is determined for $T = N$ by (10.8). The average taken over the q sample-correlations of the set determines the estimate for the sub-correlation, cf. equation (9.1).

$$\hat{R}_{N,q}(k) = \frac{1}{q} \sum_{\omega_i=1}^{q} \rho_N(k, \omega_i). \tag{11.1}$$

In a sequence of N digits there are $N - k$ products $X_j(\omega)X_{j+k}(\omega)$ equal to either $+1$ or -1. If $n_N(k, \omega)$ is the number of products equal to $+1$ then

$$\rho_N(k, \omega) = \frac{1}{N}[N - k - 2n_N(k, \omega)] \tag{11.2}$$

and

$$\hat{R}_{N,q}(k) = \frac{N-k}{k} - \frac{2}{Nq} \sum_{\omega_i=1}^{q} n_N(k, \omega_i). \tag{11.3}$$

As it has been shown in Section 10 a sample-correlation $\rho_N(h, \omega)$ is represented by a polygonal line. The vertices of this polygonal line correspond to $h = k = 0, 1, 2, \cdots, (N - 1)$ and are given by (11.2). Similarly, the estimate $\hat{R}_{N,q}(k)$, for the sub-correlation is represented by a polygonal line which is determined by (11.3). It is, therefore, sufficient to determine the values of $\rho_N(k, \omega)$ and $\hat{R}_{N,q}(k)$ at the vertices of the polygonal lines to have the corresponding sample-correlations $\rho_N(h, \omega)$ and the estimate for the sub-correlation $\hat{R}_{N,q}(h)$ for the continuous step-function. Figure 4 illustrates several examples of sample-correlation $\rho_N(h, \omega)$ for individual sequences of digits. Numerical data for $\hat{R}_{N,q}(k)$, for the five sets of sequences, are listed in Table I and a few of them are illustrated on Figure 4. The

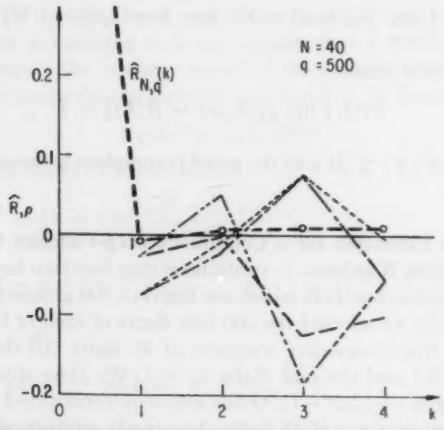

FIG. 4.—Experimental examples of sample-correlations $\rho_N(k, \omega)$ obtained for a random sequence of digits -1, $+1$, are represented using light interrupted lines. The heavy polygonal line represents the estimate of the sub-correlation $R_{N,q}(k)$ obtained by averaging over $q = 500$ sample-correlations.

theoretical sub-correlation $R_N(h)$ is in the present case equal to the correlation $R(h)$ (see (10.15)) and, therefore, the experimental results for $\hat{R}_{N,q}(k)$ should be compared to the theoretical value

$$R_N(k) = R(k) = 0, \qquad k \geqq 1. \tag{11.4}$$

The departure of the value for a sample-correlation from the estimated sub-correlation, as determined from q samples, is given by

$$\rho_N(k, \omega) - \hat{R}_{N,q}(k).$$

The variance $\sigma_{N,q}(k)^2$ for such departures is given by the relation

$$\begin{aligned} \sigma_{N,q}(k)^2 &= \frac{1}{q}\sum_1^q [\rho_N(k, \omega) - \hat{R}_{N,q}(k)]^2 \\ &= \frac{4}{N^2 q}\sum_1^q \left[-n_N(k, \omega) + \frac{1}{q}\sum_1^q n_N(k, \omega)\right]^2. \end{aligned} \tag{11.5}$$

Numerical results for the experimental variance $\sigma_{N,q}(k)^2$ are listed in Table II and compared with the theoretical values, $\sigma_N(k)^2$, computed by using the relation (10.21).

TABLE I

Estimates for the Sub-Correlations of Sequences of Random Numbers

N	40	80	120	160	200
q	500	200	100	100	100
$\hat{R}_{N,q}(1)$	−0.0041	−0.0026	−0.0047	−0.0023	−0.0053
$\hat{R}_{N,q}(2)$	+0.0053	+0.0040	+0.0143	+0.0033	+0.0054
$\hat{R}_{N,q}(3)$	+0.0066	+0.0071	+0.0087	+0.0081	+0.0073
$\hat{R}_{N,q}(4)$	+0.0083	+0.0036	+0.0095	+0.0061	+0.0072

TABLE II

Comparison between Experimental Variances $\sigma_{N,q}(k)$ and Theoretical Variances $\sigma_N(k)$

N	40	80	120	160	200
q	500	200	100	100	100
$\sigma_{N,q}(1)^2$	0.0253	0.0125	0.0094	0.0074	0.0066
$\sigma_N(1)^2$	0.0244	0.0123	0.0083	0.0062	0.0050
$\sigma_{N,q}(2)^2$	0.0231	0.0125	0.0076	0.0061	0.0048
$\sigma_N(2)^2$	0.0238	0.0122	0.0082	0.0062	0.0050
$\sigma_{N,q}(3)^2$	0.0232	0.0121	0.0074	0.0059	0.0043
$\sigma_N(3)^2$	0.0231	0.0120	0.0081	0.0061	0.0049
$\sigma_{N,q}(4)^2$	0.0198	0.0089	0.0065	0.0038	0.0035
$\sigma_N(4)^2$	0.0225	0.0119	0.0081	0.0061	0.0049

15. Frequency Distributions of Sample Correlations. Let us denote as

$$q\mathcal{F}_q[n_N(k, \omega) = u]$$

the number of sequences of a set for which $n_N(k, \omega)$ is equal to u. We find then, for the *frequency distribution of sample-correlations*

$$\mathcal{F}_q[\rho_N(k, \omega) = U] = \mathcal{F}_q[n_N(k, \omega) = u], \tag{12.1}$$

where

$$U = \frac{1}{N}(N - k - 2u).$$

Numerical values for the frequency distribution of sample-correlations can easily be obtained from Table III where $q\mathcal{F}_q[n_N(k, \omega) = u]$ is listed for the five sets of sequences. It may be noted here that (11.3) can also be written as

$$\hat{R}_{N,q}(k) = \frac{N-k}{N} - \frac{2}{N}\sum_{u=0}^{N-k} u\mathcal{F}_q[n_N(k, \omega) = u], \tag{12.2}$$

which is often more convenient for numerical computations.

TABLE III

Frequency Distributions of Sample-Correlations $\mathcal{F}_q(\rho_N = U) = \mathcal{F}_q(n_N = u)$

n_{40}	$500\,\mathcal{F}_{500}(n_{40} = u)$			
	$k=1$	$k=2$	$k=3$	$k=4$
9	0	0	0	1
10	0	0	0	1
11	4	3	2	3
12	3	3	2	12
13	8	16	10	19
14	14	13	18	23
15	23	33	27	49
16	30	45	26	44
17	49	53	46	61
18	50	51	55	74
19	58	66	69	72
20	56	70	68	65
21	68	57	59	38
22	51	29	46	18
23	38	32	28	6
24	22	9	20	8
25	11	12	13	5
26	7	4	8	0
27	2	3	0	1
28	6	1	3	0

n_{80}	$200\,\mathcal{F}_{200}(n_{80} = u)$			
	$k=1$	$k=2$	$k=3$	$k=4$
24	0	0	1	0
25	0	0	0	1
26	0	0	0	0
27	0	1	1	0
28	0	1	2	0
29	2	1	1	0
30	0	3	3	1
31	4	3	2	3
32	4	8	7	8
33	9	7	12	11
34	5	12	14	13
35	13	14	12	19
36	18	15	13	18
37	11	13	15	18
38	16	15	19	18
39	16	20	20	29
40	18	10	20	23
41	14	11	15	6
42	19	21	12	14
43	16	12	8	5
44	6	11	5	5
45	8	12	7	0
46	6	4	6	4
47	8	3	3	1
48	3	2	1	1
49	2	0	0	0
50	1	0	0	2
51	0	1	1	0
52	0	0	0	0
53	0	0	0	0
54	1	0	0	0

n_{120}	$100\,\mathcal{F}_{100}(n_{120} = u)$			
	$k=1$	$k=2$	$k=3$	$k=4$
45	0	1	0	0
46	1	0	2	2
47	0	1	2	0
48	1	2	1	1
49	2	2	1	0
50	2	0	3	1
51	1	3	0	2
52	3	4	3	7
53	6	5	4	7
54	2	3	4	7
55	7	7	10	9
56	5	13	8	8
57	6	7	9	11
58	7	7	9	10
59	5	7	9	9
60	7	6	8	4
61	6	8	2	4
62	5	7	5	3
63	4	1	6	4
64	9	4	4	4
65	6	2	3	2
66	6	3	5	0
67	1	3	0	0
68	0	2	0	0
69	3	1	0	2
70	2	0	0	2
71	0	0	0	1
72	2	0	1	0
73	0	0	0	0
74	0	1	1	0
75	0	0	0	0
76	0	0	0	0
77	1	0	0	0

TABLE III (*Continued*)

n_{180}	$100\,\mathcal{F}_{100}(n_{180}=u)$			
	$k=1$	$k=2$	$k=3$	$k=4$
60	0	0	1	0
61	0	0	1	0
62	0	0	0	0
63	1	0	0	0
64	0	0	0	0
65	0	0	0	0
66	2	2	0	2
67	0	4	2	0
68	2	1	1	1
69	2	1	1	2
70	0	3	5	3
71	1	4	3	1
72	5	4	5	3
73	6	1	3	8
74	5	3	6	4
75	3	7	8	8
76	7	4	6	14
77	6	6	6	9
78	4	4	9	7
79	6	7	5	8
80	9	12	8	6
81	4	7	1	3
82	6	5	5	6
83	1	4	6	4
84	8	3	5	3
85	4	4	4	2
86	3	2	0	2
87	2	3	2	0
88	3	4	2	1
89	1	1	2	0
90	3	0	1	1
91	0	2	1	2
92	1	1	0	0
93	0	0	0	0
94	2	0	1	0
95	1	1	0	0
96	1	0	0	0
97	0	0	0	0
98	1	0	0	0

n_{200}	$100\,\mathcal{F}_{100}(n_{200}=u)$			
	$k=1$	$k=2$	$k=3$	$k=4$
81	0	0	1	0
82	0	0	1	0
83	1	1	1	1
84	2	1	0	1
85	1	1	1	2
86	1	2	2	1
87	0	3	3	0
88	1	1	1	0
89	3	4	2	4
90	4	2	3	3
91	2	4	1	5
92	1	2	1	4
93	5	2	4	6
94	8	5	9	3
95	4	2	4	8
96	1	4	3	5
97	5	7	10	12
98	5	8	8	8
99	6	5	5	4
100	2	5	8	2
101	5	9	3	4
102	7	4	5	5
103	3	7	7	6
104	5	4	2	7
105	5	2	2	3
106	1	3	3	0
107	4	3	1	0
108	3	1	5	1
109	2	3	1	4
110	2	1	1	0
111	1	0	1	0
112	3	2	1	0
113	1	0	0	1
114	0	1	0	0
115	2	1	0	0
116	0	0	0	0
117	2	0	0	0
118	1	0	0	0
119	1	0	0	0

The theoretical probability distribution of sample-correlations is given by the relation

$$\text{Prob}\,[\rho_N(k,\omega)=U]=\frac{1}{2^{N-k}}\binom{N-k}{u},\quad u=0,1,\cdots(N-k) \tag{12.3}$$

where

$$u=\frac{N-k}{2}-\frac{NU}{2}\,.$$

For large values of $(N-k)$, this binomial distribution can be approximated by the Gaussian distribution

$$\text{Prob}\,[\rho_N(k,\omega)=U]\approx\frac{2}{\sqrt{2\pi(N-k)}}\cdot\exp\left[-\frac{(N-k-2u)^2}{2(N-k)}\right],\qquad u=0,1,\cdots N-k. \tag{12.4}$$

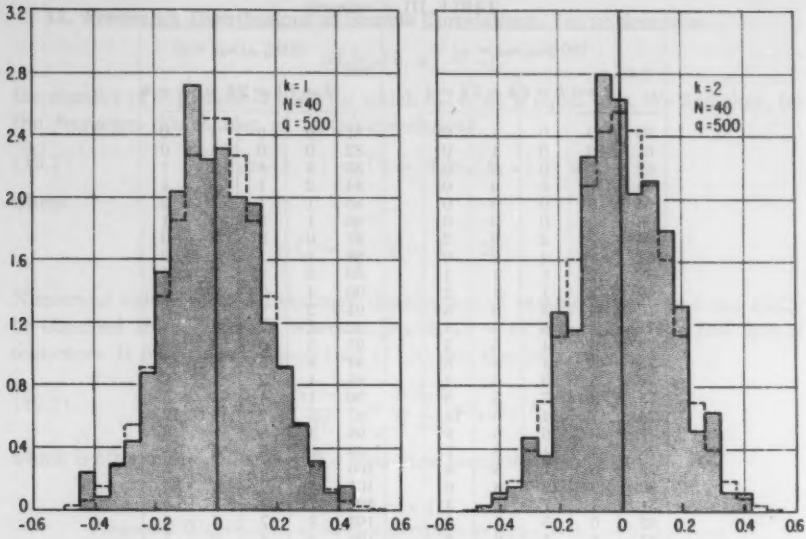

FIG. 5.—Experimental frequency distributions of sample-correlations, $\mathfrak{F}_q[\rho_N(k, \omega) = U]$ (full line) are compared with the theoretical binomial frequency distribution (dashed lines).

In particular, the probability of $\rho_N(k, \omega)$ being equal to the sub-correlation

$$R_N(k) = \overline{\rho_N(k, \omega)} = 0, \qquad k \geqq 1$$

(which, in the present case, is also equal to the correlation $R(k)$) is given by

$$\text{Prob}\,[\rho_N(k, \omega) = 0] = \frac{1}{2^{N-k}} \binom{N-k}{\frac{1}{2}(N-k)}, \qquad k \geqq 1 \tag{12.5}$$

with the approximated formula

$$\text{Prob}\,[\rho_N(k, \omega) = 0] \approx \frac{2}{\sqrt{2\pi(N-k)}}, \qquad k \geqq 1. \tag{12.6}$$

In Figure 5 we are comparing some of the experimental frequency distributions with these theoretical results.

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On Numerical Integration of Ordinary Differential Equations

By Arnold Nordsieck

Abstract. A reliable efficient general-purpose method for automatic digital computer integration of systems of ordinary differential equations is described. The method operates with the current values of the higher derivatives of a polynomial approximating the solution. It is thoroughly stable under all circumstances, incorporates automatic starting and automatic choice and revision of elementary interval size, approximately minimizes the amount of computation for a specified accuracy of solution, and applies to any system of differential equations with derivatives continuous or piecewise continuous with finite jumps. ILLIAC library subroutine #F7, University of Illinois Digital Computer Laboratory, is a digital computer program applying this method.

1. Introduction. A typical common scientific application of automatic digital computers is the integration of systems of ordinary differential equations. The author has developed a general-purpose method for doing this and explains the method here. While it is primarily designed to optimize the efficiency of large-scale calculations on automatic computers, its essential procedures also lend themselves well to hand computation. The method has the following characteristics, all of which are requisite to a satisfactory general-purpose method:

a. Thorough stability with a large margin of safety under all circumstances. (Instabilities in the subject differential equations themselves are, of course, reflected in the solution, but no further instabilities are introduced by the numerical procedures.)

b. Any integration is started with only the essential initial conditions, i.e. there is a built-in automatic starting procedure.

c. An optimum elementary interval size is automatically chosen, and the choice is automatically revised either upward or downward in the course of an integration, to provide the specified accuracy of solution in the minimum number of elementary steps.

d. The derivatives need be computed just twice per elementary step, which is the minimum consistent with controlling accuracy.

e. Any system of equations

$$\frac{dy_i}{dx} = f_i(x, y_1, y_2 \cdots) \qquad i = 1, 2, \cdots n \tag{1}$$

$$\left(\text{often written } \frac{dy}{dx} = f(x, y) \text{ for short}\right)$$

can be treated for which the f_i are either continuous or piecewise continuous functions with finite jumps.

f. The solution is computed at (although not necessarily only at) equally spaced values of the independent variable x, with specifiable spacing.

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Further useful though perhaps not indispensable characteristics of the method are:

g. Enough numerical information is developed to make interpolation or evaluation of functions (e.g., roots) of the solution possible with accuracy equivalent to the solution accuracy.

h. The sense of integration can be reversed.

Characteristic a) is essential for getting trustworthy results in lengthy automatic computations because the number of elementary steps may be as large as 10^5 or 10^6 or more, and disturbances in unstable methods typically grow exponentially with the number of steps. Characteristic b) is not only a convenience but also insures that in the integration of intrinsically unstable equations, in which early errors tend to be strongly magnified, the starting errors do not dominate. Characteristic c) relieves the human being of the often difficult task of determining the correct interval in advance. Where the human being must specify the interval for a computation not to be performed by himself he tends to make up for uncertainty by a conservatively small interval choice. Characteristics c) and d) together thus make for efficient use of computer time, and the saving in computer time can easily be a factor of 10 or even much more in the handling of problems in which the interval should vary.

In regard to the question of relating our method to previously available methods, we wish to make clear at the outset that it is equivalent to a reformulation of the method of Adams [1, p. 53–55], [2, p. 81–82] for it uses effectively the same quadrature formula as does Adams. However, the formulation and the point of view are so different that it is instructive and seems appropriate to explain the method starting from first principles, as we shall do below, rather than starting from Adams' quadrature formula.

Presently available methods may be divided into two classes: those involving no memory and those involving some memory, of the past behavior of the solution. The Runge-Kutta methods [1, p. 72–75], [2, p. 59–74] are typical of the first class, the Milne methods [1, p. 64–70], [2, p. 84] and the Adams methods of the second. It has been clear for some time that the methods with memory are superior in accuracy for a given elementary interval size and a given amount of computational labor since they permit a better approximating curve to be fitted over the elementary interval. Our method involves such memory. In return for this superiority of the methods with memory we must cope with two problems quite foreign to the memoryless methods: how to start off, since at the beginning there is nothing to remember; and how to prevent the remembered numerical information from behaving unstably.

Two further problems must be dealt with in order to implement the automatic choice and revision of the elementary interval, namely, choosing which quantities to remember in such a way that the interval may be changed rapidly and conveniently and developing an appropriate set of rules for controlling the interval size. Thus the four major problems are: automatic starting, stability, choice of quantities to remember and interval control logic. The last of these four is the most intricate.

As with most methods, there exist lower and higher "order" versions of this method. The author prefers to use the term "degree" rather than "order", since all methods are ultimately equivalent to finding a polynomial of some given *degree* approximating the solution of the system of equations, and since the term "order"

is already standardized usage for the number of equations n in (1). We have chosen and recommend degree 5, which corresponds to a truncation error $0(h^7)$ per elementary step of length h, for large-scale digital computer operations. This represents an advantageous return in accuracy per step with quite large steps, while still not overdoing the accuracy when the choice of h is limited to inverse powers of 2, as is natural in a binary computer.

The order n of the system (1) is immaterial to a large part of our discussion, so that we can advantageously use the simpler notation $dy/dx = f(x, y)$ for (1), regarding y and f as vector-like objects with n real numbers as components. The independent variable x is, of course, a single real number. Whenever the multi-component character of y and f makes a significant difference in the discussion we shall so note.

In Section 2 the choice of quantities to be remembered is discussed, in 3 the numerical procedure and the associated stability theory are developed, in 4 certain parameters of the method are adjusted for optimum stability and accuracy, in 5 the procedure for modifying the interval is given, in 6 the characteristic behavior of the remembered quantities is described, in 7 error estimation is discussed, in 8 the automatic interval control logic is developed, in 9 automatic starting is described and finally in Section 10 the results of certain test problems done by this method are exhibited. In Appendix A are collected the working formulas and error estimates for degrees 3 through 6 of the approximating polynomial. Appendix B contains a schematic flow chart for programming the method for a digital computer, with computing time estimates. Appendix C is a discussion of control of roundoff errors in iterative numerical procedures.

2. Choice of Quantities to Remember. It is immediately clear that quantities like differences $y(x) - y(x - h)$, $y(x - h) - y(x - 2h)$, etc., and/or higher differences would constitute a poor choice to remember, for changing the interval in terms of these is a cumbersome process involving much interpolation and/or extrapolation. (Ignoring the remembered quantities whenever the interval is to be changed and starting again "from scratch" would entail serious loss of accuracy and of time).

We take our cue from the remark above to the effect that all methods of numerical integration are equivalent to finding an approximating polynomial for $y(x)$. Of the many ways of specifying a polynomial of degree m by $m + 1$ constants there is one way which is interval-independent, namely: to specify the 0th to mth derivatives of the polynomial evaluated at the current value of x. These particular $m + 1$ quantities specify the same polynomial no matter what the interval is, being in fact defined with no reference to an interval at all. They would be ideal from the point of view of interval modification. However, they are not suitable for automatic computation because the higher derivatives may vary enormously in magnitude and are thus not conveniently stored in a "fixed-point" arithmetic operation.*

* The discussion in the present paper is limited to "fixed-point" arithmetic procedures. The question whether a "floating-point" version of the method could be made safe against loss or illusory gain of significance of the quantities in the course of a long computation, and otherwise trustworthy, is for future investigation. The possible freedom to store just the higher derivatives of the approximating polynomial and the increased freedom from scaling problems certainly suggest that one investigate the floating-point possibility.

In order to see how to modify our choice so as to cure the latter difficulty, we consider how the $m + 1$ derivatives would actually be used in the computation. A typical important use is, in the first phase of the integration step from x to $x + h$, to "predict" a trial value of $y(x + h)$ from the formula:

$$y^p(x+h) = y(x) + h\left\{f(x, y(x)) + \frac{h}{2!}P_s''(x) + \frac{h^2}{3!}P_s'''(x) + \frac{h^3}{4!}P_s''''(x) + \frac{h^4}{5!}P_s'''''(x)\right\} \tag{2}$$

where m has been made 5 and $P_s(x) = y(x)$, $P_s'(x) = f(x, y(x))$, $P_s'' \cdots P_s'''''$ are the 6 aforementioned derivatives of the approximating polynomial evaluated at x. Formula (2) is written in the special way shown, with one factor h external to the $\{\ \}$, because we may expect f to be computed to full register accuracy on occasion, which suggests that the remaining terms in the $\{\ \}$ be kept to the same accuracy; and because for the case of small h and many steps (many successive applications of formulas like (2)) we can minimize the accumulation of roundoff errors in y by keeping $\log(|h|^{-1})$ more places in $h\{\ \}$ than we keep in the $\{\ \}$ itself. Formula (2) in the form written then suggests that the appropriate quantities to store in the computer registers are, besides the always necessary $y(x)$ and $f(x, y(x))$, the four quantities

$$a(x) = \frac{h}{2!}P_s''(x) \quad b(x) = \frac{h^2}{3!}P'''(x)$$
$$c(x) = \frac{h^3}{4!}P_s''''(x) \quad d(x) = \frac{h^4}{5!}P'''''(x) \tag{3}$$

We may reasonably expect these quantities to stay within register capacity since an appropriate choice of h will just cause the successive terms in the $\{\ \}$ to decrease in magnitude no matter how large the $P_s^{(i)}$ themselves become. Although the quantities (3) are not completely interval-independent, they depend on the interval in such a simple way that interval change involves merely multiplying each by a constant, and in the important practical case of a binary computer and intervals restricted to inverse powers of 2 the change is achieved simply by shifting the numbers. Formula (3) seems accordingly to be essentially the unique sensible choice, at least for a fixed point arithmetic procedure.

We emphasize that the quantities y, f, a, b, c, d as they exist in the computer registers and appear in our discussion are formally defined from successive derivatives of an *approximating polynomial*, so that they always exist since an approximating polynomial always exists, whether or not the exact solution of the original problem (1) has five derivatives. If the original problem involves a discontinuous f, the quantities $a \cdots d$ tend to get large because of that, but concurrently tend to get small because of interval decrease, with the overall result that they stay within register capacity. While the existence of an approximating polynomial is assured, its quality as an approximation of the exact solution of (1) depends on how it is developed; in subsequent sections we discuss how to develop it in an optimum way.

3. Taylor's Theorem Procedure Modified for Stability. In order to have a completely defined integration procedure we must have rules for determining all of the

quantities $y(x+h)$, $f(x+h)$, $a(x+h) \cdots d(x+h)$ when given $y(x)$, $f(x)$, $a(x) \cdots d(x)$ and the differential equation (1). (The starting problem, namely to determine y, f, a, b, c, d at $x+h$ given only $y(x)$ and $f(x)$ and the differential equation, is discussed below in Section 9). Consider first the ordinary Taylor's series formulas terminated at h^5, which in terms of $a, b \cdots$ read:

$$
\begin{aligned}
y(x+h) &= y(x) + h\{f(x) + a(x) + b(x) + c(x) + d(x) + e(x)\} \\
f(x+h) &= f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x) + 6e(x) \\
a(x+h) &= a(x) + 3b(x) + 6c(x) + 10d(x) + 15e(x) \\
b(x+h) &= b(x) + 4c(x) + 10d(x) + 20e(x) \\
c(x+h) &= c(x) + 5d(x) + 15e(x) \\
d(x+h) &= d(x) + 6e(x)
\end{aligned} \tag{4}
$$

Here we have introduced one more quantity $e(x)$ analogous to $a \cdots d$, which we eliminate forthwith by using the differential equation. The system (4) as it stands is incomplete, having one less equation than it involves quantities. But by identifying the second formula of (4) with $f(x+h, y(x+h))$ calculated from the differential equation, we can eliminate $e(x)$ and get:

$$
\begin{aligned}
y(x+h) &= y(x) + h\{f(x) + a(x) + b(x) + c(x) + d(x) \\
&\qquad + \tfrac{1}{6}\,[f(x+h, y(x+h)) - f^p]\} \\
f(x+h) &= f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x) \\
&\qquad + 1\,[f(x+h, y(x+h)) - f^p] \\
a(x+h) &= a(x) + 3b(x) + 6c(x) + 10d(x) \\
&\qquad + \tfrac{1}{6}^5\,[f(x+h, y(x+h)) - f^p] \\
b(x+h) &= b(x) + 4c(x) + 10d(x) \\
&\qquad + \tfrac{2}{6}^0\,[f(x+h, y(x+h)) - f^p] \\
c(x+h) &= c(x) + 5d(x) \\
&\qquad + \tfrac{1}{6}^5\,[f(x+h, y(x+h)) - f^p] \\
d(x+h) &= d(x) \\
&\qquad + 1\,[f(x+h, y(x+h)) - f^p]
\end{aligned} \tag{5}
$$

where $f^p \equiv f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x)$, the "predicted" value of $f(x+h)$.

Now the system (5) augmented by the differential equation is complete, for the first equation of (5) and the differential equation together constitute an implicit system determining $y(x+h)$ and $f(x+h)$; the second equation of (5) is an identity and the next four then determine $a(x+h) \cdots d(x+h)$ straightforwardly.

Having arrived at the scheme (5) quite directly from Taylor's theorem we entertain the possibility of using it for numerical integration. A small amount of hand computation using (5) establishes that it is: a) very accurate indeed, and b) very unstable indeed, with small disturbances growing approximately as $(-10)^s$ in s steps.

These two phenomena are closely related. The high accuracy derives from basing the scheme directly and exactly on Taylor's theorem; however, just because it is so based it has another property, namely reversibility. If we apply (5) to go from x to $x + h$ and reapply (5) with reversed h to retrace from $x + h$ to x, we recover the original quantities $y, f \cdots d$ precisely. Now a process reversible in this sense cannot be stable, for it cannot damp out small disturbances (i.e., "forget" or "lose information") as it must to be stable. Stated in terms of the eigenvalues of the stability matrix M discussed later, reversibility implies that the matrix for backward integration is the inverse of the matrix for forward integration, which is inconsistent with the condition for stability, namely that for both these matrices all eigenvalues except one must lie inside the unit circle. (The only exception to the last statement occurs when the stability matrix is 1×1, which corresponds to the trapezoidal method $m = 1$ with no "memory.")

We search then for such a modification of (5) as will provide stability with minimum degradation of accuracy. The following discussion will establish that a usable and in fact essentially optimum modification of (5) consists of replacing the series of six coefficients 1/6, 1, 15/6, 20/6, 15/6, 1 multiplying the [ ] by new constant coefficients $Y = 95/288, 1, A = 25/24, B = 35/72, C = 5/48, D = 1/120$ respectively and leaving (5) otherwise unaltered. It is interesting to note that the ratios of the new coefficients to the old form a rather strongly decreasing sequence: 1.98, 1, 0.42, 0.15, 0.042, 0.0083, which reminds one of the well known technique for stabilizing electrical filters involving feedback by somewhat enhancing the low frequency gain and strongly depressing the high frequency gain.

In searching for an appropriate modification of (5) it is inadvisable to tamper with the coefficients *not* pertaining to the [ ], and this will be borne out by later analysis, for these coefficients are clearly just such as to make the integration of a 5th degree polynomial $y(x)$ come out exact (the [ ] will vanish for $y(x)$ a 5th degree polynomial). However, the coefficients multiplying the [ ] have no such unique significance and we are free to modify them to suit our purpose.

To dispose of the possibility of generalizing the coefficient 1 in the second equation of (5): So long as this coefficient remains 1 we can delete the second equation entirely from the considerations as being merely an identity, and we ultimately do just that. In the interests of generality the author has experimented some with modifying this particular coefficient numerically and has indeed found that any value other than 1 for it, beside costing an additional multiplication, degrades both the accuracy and the stability.

The remaining 5 equations of (5) with the coefficients 1/6, 15/6, $\cdots$ 1 replaced by arbitrary constants Y, A, B, C, D, may then be studied for stability by introducing a small variation of each of the 5 independent quantities (y, ha, hb, hc, hd), namely $(\delta y, \delta ha, \delta hb, \delta hc, \delta hd)$, and studying how this latter quintuple changes as we integrate from x to $x + h$ [3]. The quantity f is to be regarded as not independent but a function of y in virtue of the differential equation. After some calculation we find that the quintuple $(\delta y, h\delta a, h\delta b \cdots h\delta d)$, regarded as a 5-component vector $V(x)$, obeys the equation

$$V(x + h) = MV(x) \tag{6}$$

where M is a 5×5 matrix:

$$M = \begin{vmatrix} 1+\frac{p}{1-Yp} & 1+\frac{Y(p-2)}{1-Yp} & 1+\frac{Y(p-3)}{1-Yp} & 1+\frac{Y(p-4)}{1-Yp} & 1+\frac{Y(p-5)}{1-Yp} \\ \frac{Ap^2}{1-Yp} & 1+\frac{A(p-2)}{1-Yp} & 3+\frac{A(p-3)}{1-Yp} & 6+\frac{A(p-4)}{1-Yp} & 10+\frac{A(p-5)}{1-Yp} \\ \frac{Bp^2}{1-Yp} & \frac{B(p-2)}{1-Yp} & 1+\frac{B(p-3)}{1-Yp} & 4+\frac{B(p-4)}{1-Yp} & 10+\frac{B(p-5)}{1-Yp} \\ \frac{Cp^2}{1-Yp} & \frac{C(p-2)}{1-Yp} & \frac{C(p-3)}{1-Yp} & 1+\frac{C(p-4)}{1-Yp} & 5+\frac{C(p-5)}{1-Yp} \\ \frac{Dp^2}{1-Yp} & \frac{D(p-2)}{1-Yp} & \frac{D(p-3)}{1-Yp} & \frac{D(p-4)}{1-Yp} & 1+\frac{D(p-5)}{1-Yp} \end{vmatrix} \tag{7}$$

with

$$p \equiv h\frac{\partial f(x,y)}{\partial y}. \tag{8}$$

We note that the 5-dimensional vector space of V and M is a different space from the n-dimensional space of y, f, a, etc.

We have treated p as though it were a scalar quantity even though for $n > 1$ it is really an $n \times n$ matrix $h(\partial f_i/\partial y_j)$; but it is only the *smallness* of p, insurable by appropriate choice of h, which is important in our argument, not its matrix character. The difference between $p(x+h)$ and $p(x)$ has also been neglected, for it gives rise to errors involving one factor h more than we need consider.

The characteristic equation $0 = |\lambda\delta_{rs} - M_{rs}|$ of M turns out to be:

$$\begin{aligned} 0 = {} & (1-Yp)(\lambda-1)^5 \\ & + [2A+3B+4C+5D-(1+A+B+C+D)p](\lambda-1)^4 \\ & + [6B+24C+70D-(2A+6B+14C+30D)p](\lambda-1)^3 \\ & + [24C+180D-(6B+36C+150D)p](\lambda-1)^2 \\ & + [120D-(24C+240D)p](\lambda-1)-(120D)p. \end{aligned} \tag{9}$$

One root of this equation, which may be found by substituting a power series in p into it, and which we shall call the principal root λ_0, is essentially a function of p only, depending but slightly on Y, A, B, C and D:

$$\begin{aligned} \lambda_0 = {} & e^p + \frac{6Y-3+A-(1/5)C}{D}\frac{p^6}{6!} \\ & + \frac{-49+105Y+14A+(7/5)B-(14/5)C-D}{D}\frac{p^7}{7!} + 0(p^8). \end{aligned} \tag{10}$$

This is a consequence of retaining the coefficients in (5) not pertaining to the [ ]. The root λ_0 is thus essentially a property of the differential equation system (1), and whether or not it lies inside the unit circle in the complex λ plane determines

whether the subject system, as distinguished from our numerical method, is stable or not. On the other hand, the four further roots of (9), which we shall call "extraneous" roots, depend strongly on A, B, C, D and only weakly on p and Y; their location relative to the unit circle determines the stability of the integration method itself. These roots must lie inside the unit circle for stability of the method, and the nearer they are to the origin the more stable the method will be.

4. Determination of Parameters. The parameters $Y \cdots D$ are now to be chosen, primarily to optimize the stability of the method and secondarily, if any freedom is left over, to optimize the accuracy within the restriction of optimum stability. The author regards optimum stability as essential to an automatic general-purpose method, for the rapid elimination of disturbances characteristic of good stability not only makes an automatic starting process feasible and permits accurate integration across finite discontinuities of f, as we shall see below, but also minimizes the error due to interaction of disturbances with non-linearities of the differential equations.† Since there are four extraneous eigenvalues whose locations in the complex plane we wish to control and we have five parameters free, we can expect to have considerable control over stability and accuracy. What actually happens is that A, B, C, D determine stability and Y is left free to optimize accuracy. Thus we can arrange for a truncation error of $0(h^7)$ even though we are using 5th degree polynomials, the explanation being that in each integration step we use both the 5th degree polynomial available at the beginning and the one available at the end of the step.

Now it is easy to bound $|p|$ (bound the magnitudes of its eigenvalues if it is a matrix) by control of h during the numerical integration process, while it is much more difficult actually to compute p for $n > 1$. Therefore it seems best and is certainly simplest to choose Y, A, B, C, D independent of p, i.e. as absolute constants, in such a way that stability is guaranteed for as large a range of p as possible. This is substantially accomplished by considering (9) with $p = 0$ (whereupon Y drops out, indicating that it has little influence on the stability of the method) and then choosing A, B, C, D so that the four extraneous roots coincide at 0. Thus, we require (9) for $p = 0$ to take the form $(\lambda - 1)\lambda^4 = 0$, and it does that for $A = 25/24$, $B = 35/72$, $C = 5/48$, $D = 1/120$. The choice of Y is then made to nullify the coefficient of p^6 in (10), which has no effect on the stability but optimizes the accuracy. This determines $Y = 95/288$. For stability for $p \neq 0$ we then depend on the fact that the extraneous roots are continuous functions of p, so that they cannot move very far from the origin provided p is appropriately limited.

In order to get a better picture of the behavior of the extraneous roots as functions of p, we first note that for small p they are the roots of

$$\lambda^4 = -\frac{3}{160}\,p \tag{11}$$

as can be read off from (9) with the chosen values of the parameters inserted. It is

† A report by E. Fehlberg [4], has just come to the author's attention. Fehlberg exhibits other choices of parameters which produce smaller truncation error than Adams' and the author's choice, *but* at the expense of much poorer stability, cf. Fehlberg's tables 3 and 4. For $m = 5$ the gain in computing speed for the same error is greatest and is $(1/0.0801)^{1/7} = 1.43$, which the author considers not worth the risks incurred with the much poorer stability.

fortunate that the numerical coefficient in (11) is so small, for the $p^{1/4}$ dependence of the roots is a rather strong dependence. The roots have also been computed for p a real number between -1 and $+1$, and these are shown in Figure 1. We see that stability will be guaranteed with a comfortable margin of safety if the interval is so chosen that p lies effectively inside the dashed curve. This boundary corresponds to $| Yp | \leqq 1/8$, which is a convenient form of test for a computer.

The author has done considerable searching for other favorable choices of A, B, C, D with the thought in mind that if the extraneous roots never coincided they might move away from the origin more slowly as $| p |$ increased, than they do according to (11). However, all other choices tried were inferior in point of both stability and accuracy.

The choice of parameters made above seems optimum among choices restricted to constants independent of p. The potential advantage of a more elaborate procedure in which the matrix p is numerically computed at every step and $Y, A \cdots, D$ are made chosen functions of p, implying a nonlinear process tailored to the subject differential equation system, is an interesting topic for future investigation, for it might lead to faster (though less accurate) methods of solving some classes of equations.

The working equations of the method have now been determined completely and they are summarized in Appendix A, equations $(A4)$.

The working equations having been determined, the precise connection with

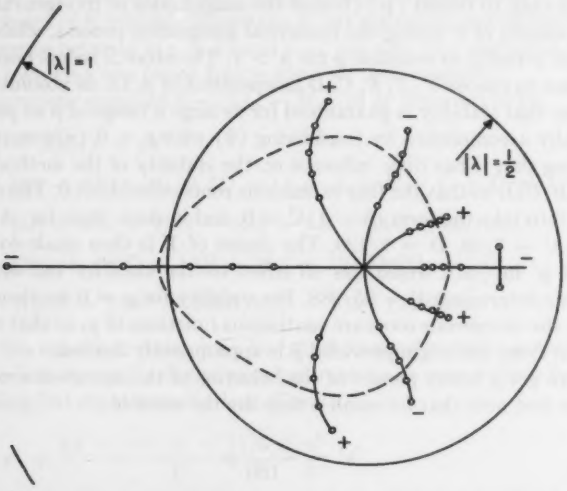

FIG. 1.—The extraneous roots of the characteristic equation as functions of p for real p, plotted in the complex λ plane. As p departs from zero these roots depart from the origin along the loci shown. Loci marked + correspond to positive p and loci marked − to negative p. Counting outward from the origin along each locus, the points plotted represent in order, $|p| = \frac{1}{16}, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1$. The real positive extraneous root coalesces with the principal root at $p = -0.88$, producing a conjugate pair. The dashed curve encloses all extraneous root values permitted by the interval control tests, which limit $|p|$ to values $\leqq .38$.

other methods can be deduced by ascertaining the equivalent quadrature formula for the method. This can be done by expressing the part $h\{\ \}$ of the first working equation in terms of past values $f(x-h), f(x-2h)$, etc., by repeated application of the working equations. We find that the equivalent quadrature formula is

$$y(x+h) - y(x) = \frac{h}{1440}\{475f(x+h) + 1427f(x) - 798f(x-h) + 482f(x-2h) - 173f(x-3h) + 27f(x-4h)\} \tag{12}$$

which agrees exactly with the Adams formula of corresponding degree. By way of confirmation of this conclusion we observe that the characteristic equation for small variations in the Adams method coincides with (9) when the chosen values of the parameters are inserted into (9).

5. Change of Interval. We indicate how to perform the three useful changes of interval: $h' = -h$, $h' = \beta h$ and $h' = \beta^{-1}h$ (where in binary computer operations β is preferably taken equal to 2):

Reversal	Increase	Decrease	
$-h$	βh	$\beta^{-1}h$	replaces h
y	y	y	" y
f	f	f	" f
$-a$	βa	$\beta^{-1}a$	" a
b	$\beta^2 b$	$\beta^{-2}b$	" b
$-c$	$\beta^3 c$	$\beta^{-3}c$	" c
d	$\beta^4 d$	$\beta^{-4}d$	" d

(13)

The rules for changing a, b, c, d are clear from (3).

The simplicity of the rules for changing the interval is evident here.

Every change of interval of any of the three types induces a disturbance in the system, but the disturbance affects mainly the higher derivatives and clears out in a few steps because of the choice of parameters. These transient phenomena will be described in more detail in the next following section.

6. Behavior of a, b, c, d. A qualitative understanding of the behavior of the quantities constituting the method's "memory" is required in order correctly to design the interval control logic and the starting procedure.

We first describe the "normal" or steady behavior which prevails when no transients have been induced by interval change or f-discontinuity or otherwise, within the preceding 4 to 8 steps or so. Then the quantities a, b, c, d "lag" behind the current value of x, a a little, b more, c still more, and d most, in the sense that they equal the "true" higher derivatives of y evaluated at points $x - \theta h$, where $0 < \theta \leqq 2$. This lagging behavior is related to, and is in fact a necessary consequence of stability. A close analogy exists between this and the "stable physically realizable filter" of electrical engineering theory, and likewise the causality discussions in physics. The indicated behavior may be established (and incidentally some formulas of later use for deriving the truncation error found) by assuming that a 7th degree polynomial $y = P_7(x)$ satisfies the differential equation exactly and that f, a, b, c, d are corresponding polynomials of 6th $\cdots$ 2nd degree, and solving the working

equations (A4) for the coefficients by some rather lengthy algebra. The result is

$$
\begin{aligned}
a(x) &= \frac{h}{2!}\,y''(x) - 72\,\frac{h^5}{6!}\,y^{VI}(x) + 840\,\frac{h^6}{7!}\,y^{VII}(x)\\
b(x) &= \frac{h^2}{3!}\,y'''(x) - 100\,\frac{h^5}{6!}\,y^{VI}(x) + 1110\ 2/3\,\frac{h^6}{7!}\,y^{VII}(x)\\
c(x) &= \frac{h^3}{4!}\,y''''(x) - 52\ 1/2\,\frac{h^5}{6!}\,y^{VI}(x) + 525\,\frac{h^6}{7!}\,y^{VII}(x)\\
d(x) &= \frac{h^4}{5!}\,y'''''(x) - 12\,\frac{h^5}{6!}\,y^{VI}(x) + 91\,\frac{h^6}{7!}\,y^{VII}(x).
\end{aligned}
\tag{14}
$$

These formulas are then in error by $0(h^7)$ for any general y which is differentiable sufficiently many times. The last of the four formulas shows that $d(x) = \frac{h^4}{5!}\,y'''''(x-2h) + 0(h^6)$, so that d lags by very nearly two steps.

We may describe this "normal" behavior in another way, namely, by observing that the polynomial evaluated at x is always essentially the polynomial fitted to the values of y at $x, x-h, x-2h, x-3h, x-4h$. The 5th derivative of this polynomial naturally agrees best with the 5th derivative of the true solution y at the mid-point of the fitting interval, which explains the last equation of (14). The close relation of our method to the Adams method also becomes clear from this point of view. When we advance from x to $x+h$ the working equations in effect change the old polynomial fitted at $x-4h \cdots x$ into one fitted at $x-3h \cdots x+h$. In the approximation that the 4th powers of the extraneous characteristic roots may be neglected, all disturbances clear out in precisely 4 steps, corresponding to the memory of the method having a "time-span" of just 4 steps. Thus, we have arranged *effectively* to keep and use what Adams actually keeps and uses, namely the last four previous ordinates, whereas actually we keep quantities much more suitable for interval modification.

As for "abnormal" behavior of the remembered quantities, the simplest important case of this occurs upon reversal. The quantities exhibit a hysteresis after reversal, most pronounced in the case of $d(x)$ which has the most lag. The behavior of d in reversal is illustrated in Figure 2, which shows essentially that d stays quite strictly constant for four steps after a reversal, then abruptly resumes normal behavior. Since what was a backward-fitted polynomial before reversal becomes a forward-fitted polynomial after reversal, we may say that d and indeed the polynomial as a whole "freezes", remains the same and marks time until enough steps have been executed for it to become normal for the current point x, then behaves normally.

The other important type of abnormal behavior is the response to shock excitation. Shock excitation occurs severely in starting, when the normal $a \cdots d$ are not known; mildly enough to be harmless in increasing or decreasing the interval, when the main terms but not the "lag" terms in (14) are correctly modified by the simple rules (13); and more or less severely when f has a discontinuity so that the change in the polynomial is large in one step. Here again $d(x)$ shows the most violent behavior and its behavior in all shock-excited transients is essentially an oscillation lasting just four steps.

TABLE 1

x	y	f	$24a$	$72b$	$48c$	$120d$
x_0	0	0	0	0	0	0
$x_0 + h$	$\left(\frac{1}{2} - \frac{245}{1440}\right)h\Delta$	Δ	25Δ	35Δ	5Δ	Δ
$x_0 + 2h$	$\left(\frac{3}{2} + \frac{217}{1440}\right)h\Delta$	Δ	-23Δ	-69Δ	-13Δ	-3Δ
$x_0 + 3h$	$\left(\frac{5}{2} - \frac{119}{1440}\right)h\Delta$	Δ	$+13\Delta$	$+45\Delta$	11Δ	3Δ
$x_0 + 4h$	$\left(\frac{7}{2} + \frac{27}{1440}\right)h\Delta$	Δ	-3Δ	-11Δ	-3Δ	$-\Delta$
$x_0 + 5h$	$\frac{9}{2}h\Delta$	Δ	0	0	0	0

In order to become familiar with the detailed behavior of such a transient, we treat a simple case which approximates the general case of an isolated discontinuity of f with finite jump: Let $y \equiv f \equiv 0$ for $x \leqq x_0$ and assume that $a \cdots d$ have their normal values of zero for $x \leqq x_0$. Let $f \equiv \Delta =$ constant for $x > x_0$ and apply the working equations (14) five times in succession and Table 1 results. Evidently $\frac{2!}{h}a, \frac{3!}{h^2}b, \frac{4!}{h^3}c$ and $\frac{5!}{h^4}d$ are behaving like numerical approximations to the "δ-function" of x and its first, second and third derivatives respectively. Meanwhile the transient in y, represented by the terms with denominator 1440, is a decreasing oscillation also lasting just four steps, and the ultimate value of y is exactly what one would get by connecting the last point sampled at which $f = 0$ with the first point sampled at which $f = \Delta$ by a straight line segment. This essentially best performance in integrating across a discontinuity is unique to our choice of parameters. A reasonable upper bound for the magnitude of the error in y due to such a discontinuity is $\frac{1}{2} \mid h\Delta \mid$ where Δ is the jump in f. One can hardly do better without sampling in between these two points, i.e. decreasing h; but by controlling h one can bound this error.

7. Estimation of Errors. In discussing errors in the solution $y(x)$ we must distinguish between the error present at the beginning of an elementary step and the error contributed by the execution of that step. The error present at the beginning of a step, sometimes called inherited error, is the net result of all the errors contributed by all the previous steps, each modified according to the action of the differential equation between the point of origin x' of the error and the current point x. Letting $E(x')$ represent the error contributed by a step of length h taken at x', we can write for the inherited error at x if the integration began at x_0:

$$E_i(x) = \sum_{x'=x_0}^{x} E(x') \prod_{x''=x'}^{x} \lambda_0(x'') \tag{15}$$

where $\lambda_0 \cong e^{\theta}$ is the principal root (10). The sum involves the summand once for every elementary step taken and similarly the product. Equation (15) illuminates the relation between inherited error and error contributed by an individual step.

The product in (15) may also be written in adequate approximation as:

$$\prod_{x'}^{x} \lambda_0(x'') = \exp\left\{\int_{x'}^{x} \frac{\partial f}{\partial y}(x'')\, dx''\right\} \tag{16}$$

which shows that the product is a property of the differential equation, independent of integration method and of interval choice. It is clear then that although by careful design of the method and choice of interval we may be able to reduce $E(x')$ down to about half the least count in the register (but no further because of inevitable rounding), nevertheless such measures have no effect on (16). Consequently if Λ is the largest eigenvalue of the matrix (16) the error at the conclusion of the integration will be in general at least about $\frac{1}{2} \mid \Lambda \mid$ times the least count. The number of correct significant digits may at most be preserved through the calculation if the magnitude of the solution increases by Λ or more; if not, the significance (i.e. the number of correct significant digits) will decrease. If a problem has $\Lambda > \beta^L$, where L is the number of base β digits in the register, then it is useless to attempt the problem at all by fixed point arithmetic, for there will be no correct significant digits left at the end of the calculation. Floating point could help if the magnitude of the solution increases meantime; if not, nothing will help except increased register length.

We have dwelt on the above points because they show that the best that can be done with any method is approximately to preserve the number of correct significant digits in the solution, and this essentially defines a best or optimum method. Some of the test examples exhibited in Section 10 below show nearly complete preservation of significance through as many as 10^5 steps and with Λ as large as 10^6 or so.

Turning now to discussion of $E(x)$, we assert that the contributions to $E(x)$ are: a) truncation error incurred by terminating the formulas (A1) to (A5) with a given power of h; b) discontinuity error incurred in integrating past a discontinuity of f (cf. Section 6); c) iteration error resulting from incomplete iterative solution of the implicit equations for $y(x + h)$; and d) roundoff error resulting from using registers of finite length to perform the arithmetic.

The truncation error may be found by making the same assumptions $y = P_7(x)$ etc. as were made in deriving equations (14) and calculating $y(x + h) - P_7(x + h) - y(x) + P_7(x)$, using the first and second working equations and (14). We find that the truncation part of E, which we call E_t, is given by:

$$E_t(x + h/2) = 72 \frac{h^7}{7!} y^{VII}(x) + 0(h^8). \tag{17}$$

It is interesting to note that the truncation error is closely related to the principal root of the stability matrix. In fact, if we replace p arbitrarily by the operator $h\frac{d}{dx}$, $\left(\text{because the proof that } p \text{ is precisely equivalent to } h\frac{d}{dx} \text{ is not apparent}\right)$, then e^p becomes the "true" displacement operator $e^{h(d/dx)}$ and $\lambda_0(p)$ becomes the approximate displacement operator of the method. Thus the difference $\lambda_0(p) - e^p$ with p replaced by $h\frac{d}{dx}$, and applied to $y(x)$, would seem to yield the truncation error. The term in h^8 in the truncation error was determined by exploiting this relationship,

yielding:

$$E_t(x + h/2) = 72\frac{h^7}{7!}y^{VII}(x) - 440\frac{h^8}{8!}y^{VIII}(x) + 0(h^9) \tag{18}$$

The discontinuity error, called E_d, is bounded by the inequality

$$|E_d(x)| \leqq \tfrac{1}{2}|h(f(x_+) - f(x_-))| \tag{19}$$

as we saw in Section 6.

The iteration error, called E_i, depends on how we solve the implicit equation system, and we choose to solve it by doing just two iterations, or more precisely: We calculate a first trial value $y^{(1)}(x + h)$ from equation (1) of (A4) with the [ ] term left off (the "predicted" $y(x + h)$ in Milne's terminology); calculate $f^{(1)}(x + h) = f(x + h, y^{(1)}(x + h))$ and insert it on the right of the complete equation (1) of (A4) to give an improved $y^{(2)}(x + h)$; and repeat the procedure just once more, so that by definition in this method the final values of $y(x + h)$ and $f(x + h)$ are $y^{(2)}(x + h)$, respectively $f(x + h, y^{(2)}(x + h))$. The reasons for choosing so are that f need be calculated only twice, that the convergence of the iterative procedure and the (related) bounds on p can be estimated from two iterations but not from less than two, and that the iteration error is sufficiently small. For the special case $n = 1$ (a single first-order differential equation) one can do better by solving the implicit system by interpolative methods with the same number of computations of the derivative; for general n, however, one would have to compute the derivatives $2n$ times at least in order to apply interpolative methods, which we regard as uneconomical. The convergence is determined by the equation

$$y^{(3)} - y^{(2)} = Yp(y^{(2)} - y^{(1)}); \qquad Y = \frac{95}{288} \tag{20}$$

and the "iteration error" in $y(x + h)$ by

$$E_i = y^{(3)} - y^{(\infty)} = (Yp)^2(y^{(1)} - y^{\infty}) \cong -Y^3p^2h^6y^{VI}(x) \tag{21}$$

which is proportional to h^8 with a small coefficient so long as $|Yp| \leqq \frac{1}{8}$ as we shall require, and is therefore overshadowed in general by the truncation error. Equations (20) and (21) follow from iterative treatment of equation 1 of (A4).

The roundoff error E_r, finally, is determined by the care with which both the computation of derivatives and the computations of (A4) are done, and with sufficient care can be as small as about $\frac{1}{2}$ the least count in the effective register used and approximately statistically independent from step to step. The author has found it best to keep $\log_2(|h|^{-1})$ extra "guard" digits in y, above and beyond the number kept in $f, a, \cdots d$, in order to minimize the accumulation of roundoff errors in y when the number of elementary steps is large.

8. Automatic Interval Control Logic. In order to describe the interval control we must first outline the 3 stages in which a step $x \to x + h$ is performed. Stage 1 consists of "predicting" all six quantities $y, f, \cdots d$ at $x + h$, i.e. applying equations (A4) without the [ ] terms, using a tentative value of h. The first tentative value of h actually tried is the value which was accepted in the last previous step or the next larger value if the conditions (given below) for increasing h were fulfilled. Note that Stage 1 is exactly reversible in a digital machine, so that if h later turns out to

be wrong the beginning values of $y \cdots d$ can be exactly recovered without the need for additional registers for saving them. Stage 2 consists of solving the implicit equation system for $y(x+h)$ and $f(x+h)$ by iterating twice as explained in the preceding section. This stage is not exactly reversible and $2n$ registers are therefore provided for saving the beginning values of y and f. At the conclusion of Stage 2 enough information has been developed to decide whether the interval tentatively being used is small enough; if it turns out to be not small enough the beginning values of $y \cdots d$ are recovered, the interval is reduced (by a factor $\beta^{-1} = \frac{1}{2}$ in a binary computer) and Stage 1 is again entered. If the tentative interval is found adequate we proceed to Stage 3, which consists of "correcting" a, b, c, d by adding the [ ] terms.

Two tests are made at the conclusion of Stage 2 and failure of either signifies that h is too large; the two tests are respectively

$$| y_i^{(3)} - y_i^{(2)} |_{\max} \leqq \tfrac{1}{6} | y_i^{(2)} - y_i^{(1)} |_{\max} \tag{22a}$$

and

$$| f_i(x+h) - f_i^{p} |_{\max} \leqq \beta^{-e} / | h | \tag{22b}$$

where e is a specifiable positive integer and "max" means the largest of the n components $i = 1, 2, \cdots n$. It is clear that these tests are first possible at the end of Stage 2, since they involve quantities developed only in that stage. While the tests are being made it is also determined whether both tests are "over-satisfied", i.e. so well satisfied that the next larger h would likely also satisfy them, and if so the interval may be tentatively increased for the next following step.

Satisfying test (22a) insures that the largest eigenvalue of p does not exceed 0.38 in magnitude (cf. equation (20)) and, therefore, that the stability is good (cf. Figure 1) and also that the iteration error is small enough to be overshadowed by the truncation error (cf. equation (21)). The test is not formulated in the ideal way, which would be to require the Euclidean norm of the difference vector to decrease by $\frac{1}{6}$; instead we require that the largest component of the difference vector decrease by at least $\frac{1}{6}$, which is equally effective in insuring convergence, works for any order n, and requires less computation and less registers.

Satisfying test (22b) then has the effect of roughly bounding the truncation error and the discontinuity error in such a way that the *accumulated* error in integrating a standard distance Δx (which we take equal to 1) is independent of the elementary step-lengths used and about equal to β^{-e}. In effect, instead of having to specify the elementary step-lengths to be used, the programmer tells the computer he wants the eth digit in y to be correct after integrating a unit distance along the x axis and the computer is expected to choose the elementary intervals to achieve this result most economically. Note, however, that in this connection the discussion of preservation of significance for unstable equations given at the beginning of Section 7 must be kept in mind.

Test (22b) is derived from equation (17) by the following rough argument. We divide the interval $(x_0, x_0 + 1)$ into subintervals in such a way that within each subinterval h is constant. Then summing (17) over the kth subinterval gives:

$$\mathcal{E}_k \equiv \sum_{x=x_k}^{x_{k+1}} E_i(x) \cong \frac{h^6}{70} \int_{x_k}^{x_{k+1}} y^{VII}\, dx \cong \frac{h^6}{70} (y_{k+1}^{VI} - y_k^{VI}) \tag{23}$$

Now the computation provides an estimate of h^6y^{VI}, namely, $h[f(x+h) - f']$, as may be deduced from the 6th equation of (A4). We use this estimate to bound h^6y^{VI} for all x by requiring satisfaction of test (22b) in every elementary step. Thus $h^6 \mid y^{VI} \mid \leqq \beta^{-e}$ and the accumulated error is, roughly speaking, bounded by

$$(24) \qquad \mid \sum_k \varepsilon_k \mid \leqq \frac{\text{no. of subintervals}}{70} \cdot \beta^{-e}$$

which is not likely to be much greater than β^{-e}. We see also that the general effect of bounding h^6y^{VI} is to cause each part of the total integration interval to contribute to the error in proportion to its length, which tends to minimize the total number of steps to achieve a given accumulated error. The argument is necessarily somewhat crude, for we cannot do what one would ideally like to do, namely, bound h^6y^{VII}, because there is no estimate of it available (without increasing the degree of the method). Test (22b) also bounds the discontinuity error, equation (19), for a discontinuity if f clearly appears directly in $[f - f']$, so that bounding $h[f - f']$ just bounds (19).

In addition to availability of an estimate there is a further practical reason for formulating test (22b) in just the way shown, at least in a fixed point arithmetic operation, namely, that it permits the widest possible range of choices of h without either member of the inequality falling outside register range. If one wants to integrate across large discontinuities of f and still be free to demand accuracy of the order of the least count, it is clear from (19) that h must be reducible to or near the least count; on the other hand, for maximum size steps when f varies slowly and smoothly h must be increasable to or near the greatest count of the register. In practice the author has had the interval vary all the way from 2^{-2} to 2^{-30} in a 39 binary digit machine.

In the main then, the interval is selected by requiring it to be the largest interval satisfying both tests (22a) and (22b). However, four minor modifications of this basic rule are introduced in order to improve the usefulness and efficiency of the method and the smoothness of the automatic interval control, as follows:

Since the programmer cannot predict what intervals will be used he is given the privilege of specifying a maximum interval h_0, so that he has assurance that the solution will be available at least at the points $x_0 + (\text{integer})h_0$. The automatic interval control then includes a feature preventing an increase in the interval whenever such increase would result in skipping over one of the above points $x_0 + (\text{integer})h_0$.

Next, when any considerable amplitude of shock excitation has occurred it seems best, judging from Table 1, to choose the interval at the onset of the shock, then leave it unchanged until the transient due to the shock has subsided. In fact, if the interval is changed while the strong transient is still present this interval change itself results in a new shock excitation, and the interval control tends to become erratic in the sense that the interval is reduced too much and for too long, a phenomenon which the author has observed experimentally. The interval control itself contains feedback loops, we may say, which can cause erratic behavior, although not genuine instability because the computer takes refuge in *reducing* the interval in response to any uncomfortably large disturbance. The main rule, if not modified, leads to just such behavior because, as we see from the last column of Table 1

the change in d is 4 to 6 times greater in steps subsequent to the first step after onset of the disturbance than in the first. To avoid this misbehavior the computer is programmed to recognize the characteristic Δ, -4Δ, $+6\Delta$, $-4\Delta \cdots$ pattern and to leave the interval unchanged on the 2nd, 3rd and 4th steps provided they conform to this pattern within certain tolerances. This effectively prevents the interval control from interfering with the expeditious elimination of transients and results in preserving the ideal accuracy and speed represented by Table 1.

Another form of undesirable interference from interval control occurs in connection with reversal. Suppose that reversal has just occurred and that test (22b) is dominant in determining the interval, as it often will be. From Figure 2 we see that just after reversal d stays constant for 4 steps. This means that (22b) will be oversatisfied and the interval will be increased, whereas it should clearly not be increased since we are retracing steps for which the interval was presumably already correctly chosen earlier. The subsequent behavior would involve an unusually large shock when the "slack" in d is eventually "taken up" and an unnecessarily large interval decrease, again a phenomenon the author has observed in practice. The remedy for this misbehavior is simple: we program in a rule preventing interval increase for the first four steps after any reversal.

Finally a rather interesting type of misbehavior can occur when f tends toward a constant or indeed toward any 4th degree polynomial after an earlier more violent behavior which required a small interval. In these circumstances we want and expect the interval to increase rapidly, but if the parameter β^{-e} of test (22b) is very small, say only a few times the least count, then such increase may be prevented entirely by persistent roundoff noise in the "remembered" quantities. If f tends asymptotically to a 4th degree polynomial d should tend to a constant and $[f(x+h) - f^{[}]$ should tend to 0. What happens then is that so long as roundoff noise persists either (22b) is barely satisfied and the interval is not increased, or if (22b) is oversatisfied and an interval increase is attempted the roundoff noise in d is magnified by a factor β^4 according to (13) and causes test (22b) to fail on the next step. Now, unless special measures are taken, the roundoff noise can indeed persist and prevent interval increase indefinitely. Thus we may get into (and the author has actually got into) the absurd situation of taking 4000 steps to integrate $\frac{dy}{dx} = 0$ from $x = \frac{1}{2}$ to $x = 1$ (provided f was non-zero for $x < \frac{1}{2}$). The remedy for this mis-

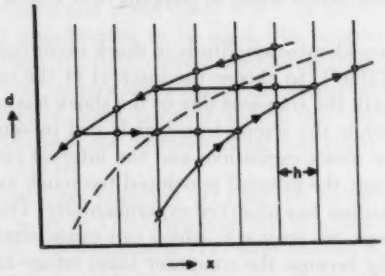

FIG. 2.—Hysteresis behavior of the "remembered" quantity d. The dashed curve is the true value of $d(x)$; the solid curves show the behavior of d in the computation.

behavior is not modification of the rules for interval choice, but a peculiar, carefully chosen rounding procedure for the multiplications by $A, \cdots D$ involved in the working equations so as to guarantee that roundoff noise will disappear in a finite (and minimum) number of steps just as other transients must and do because of the stability. The discussion of choice of rounding is rather long and is also of interest for other iterative procedures in numerical analysis, therefore, it is given separately in Appendix C.

The main rule amended by the four modifications just described provides a stable, non-erratic and generally reasonable behavior of the interval size in all cases which have been investigated, and the cases investigated were purposely chosen extremes in which the interval had to vary rapidly and widely. The interval still does not increase as fast when it should increase as it decreases when it should decrease, but this is hardly avoidable since both the finite rate of clearing of transients and the requirement of not skipping over the points $x_0 + (\text{integer})h_0$ act to delay interval increase.

If, when h has been reduced to the least count, test (22b) still fails, a programmed stop is encountered. Almost any major malfunction of program such as overflow in the computation of derivatives or elsewhere leads quite immediately to this programmed stop because of the extreme sensitivity of test (22b).

9. Automatic Starting. The essential idea which makes automatic starting feasible is that if we set off with entirely abnormal values of a, b, c, d, say putting 0 for each of them in the absence of any evidence as to their normal initial values, then upon integrating a few steps they will assume approximately their normal values *if* the stability is sufficiently good. Such a method of starting has the advantage of using mostly the normal integrating program, which has to be supplied in any case, requires very little extra programming of special nature, is of use only during starting. Since a modern computer can execute at least about one step per second in even rather complicated differential equation problems, the start can be accomplished blunder-free and accurately in a matter of seconds or at most minutes.

Several complications must be dealt with in providing a satisfactory automatic start: the proper interval for the first step forward from x_0 is not known in advance any more than are a, b, c, d. There is a certain degree of incompatibility between automatic starting and interval changing since the starting essentially involves eliminating a very large transient and, as we saw in the preceding section, changing the interval during a large transient can lead to erratic interval behavior. In any case, application of test (22b) during the first few steps of the starting process would be meaningless since the test was derived on the assumption that $a \cdots d$ had nearly normal values; this is illustrated by the fact that when a, b, c, d are zero the quantity $[f(x + h) - f']$ is $0(h)$, not $0(h^5)$ as it normally is. Finally, although one would ideally like to use points to the left of x_0 for starting, corresponding to fitting a polynomial to the left of x_0 and thus obtaining what we have called normally lagging values of $a(x_0) \cdots d(x_0)$, this cannot be done because it would imply that f is defined to the left of x_0, as it may not be.

The detailed schedule of the starting procedure will now be described and in the process the way in which the complications listed above are dealt with will become clear. The overall objective of the starting procedure is to fit a 5th degree poly-

nomial for y to the points x_0, $x_0 + h/2$, $x_0 + h$, $x_0 + 3h/2$, $x_0 + 2h$, thus determining $a(x_0)$, $b(x_0)$ $\cdots$ $d(x_0)$, where h is the correct interval (also to be determined) for the first step $x_0 \rightarrow x_0 + h$.

First we set the initial values $y(x_0) = y^0$ aside for safekeeping, set $a \cdots d$ equal to zero and do a tentative step forward $x_0 \rightarrow x_0 + h_0$, where h_0 is the maximum interval permitted. Test (22a) (but not (22b)) may now be applied since its operation is essentially independent of whether $a \cdots d$ have their normal values. If (22a) fails the interval is reduced, the beginning values at x_0 are recovered and a shorter tentative step forward from x_0 is taken, the program used here being just the same as in normal integration. This process continues until an h has been found which satisfies (22a).

When (22a) has been satisfied three more steps forward are taken, followed by a reversal and four steps back to x_0, all eight steps being taken at a constant interval. The reason for taking just four steps either way is that it provides just enough information to determine a 5th-degree polynomial.

We are now back at x_0 with a value of y somewhat in error but with first approximations for $a \cdots d$ which are already good to a fraction of a percent because of the high degree of stability of the method. The correct value of $y(x_0)$ is reinserted, the sense of integration again changed to forward and another four steps forward and four steps back to x_0 are taken, all at the same constant interval.

During the last backward step listed (the 16th step of the starting process) test (22b) is activated, for now the quantities $a \cdots d$ are so nearly normal that this test is significant. Test (22b) must be made neither too early during the starting process, for then $[f(x + h) - f^p]$ is not yet $0(h^5)$; nor too late, for as the process of integrating four steps back and forth is continued, $[f(x + h) - f^p]$ tends to *zero* in any case (refer to the hysteresis behavior of d described in Section 6). Thus there is a sort of psychological moment for doing test (22b) during the starting process. The author has found by "experimental mathematics" that $[f - f^p]$ is 2 to 3 times larger on the 16th starting step, for all equations and all h's, than it is in the ultimate normal integration process. Thus, applying the test at this point results in a slightly conservative initial choice of h.

If (22b) is not satisfied the interval is reduced and we go back to the very beginning of the starting process. If (22b) is satisfied, y^0 is reinserted, the sense of integration is changed to forward and the starting process may be considered almost completed. In fact, for all cases except those with very high accuracy requirements and very unstable equations the above process provides a satisfactory start. In the exceptional cases one can do a little better, typically a factor of six in the initial truncation error, by extending the starting schedule to include four more steps forward and four back at *half* the interval eventually to be used (making 24 starting steps in all after both tests are satisfied) and we actually take these extra eight steps in order to be quite sure that errors attributable to starting are less than the normal running truncation error. More precisely, after test (22b) is satisfied during the 16th starting step, we reinsert y^0, change the sense to forward, halve h, integrate forward four steps, reverse, integrate back four steps to x_0, reinsert y^0, change the sense to forward, double h and *now* regard the starting process as complete.

The chief effect of performing the last eight starting steps at a reduced interval is to reduce the amount of lead in a, b, c, d, which is beneficial because they should

lag, as they would if we had used a backward fitted polynomial. In any event, the truncation error in the first step after starting as above is less than normal.

Note that we have avoided ill effects due to changing interval during a transient by insisting that if any starting step at all is taken with a given interval h, at least eight are taken without changing h.

The transients during the early stages of the starting process are often large enough to cause overflow of the computer registers, and it is interesting to observe that such overflow will do no harm, for test (22b) is a very sensitive test and will almost certainly be violated if there are any previously occurring overflow errors. When this test is violated the computer simply discards all its previous computations, including any overflow errors, and starts afresh with reduced interval. The author has observed this effect many times, always without ultimate consequences. Persistent overflow caused by incorrect scaling of x, y or f is of course another matter, but one which comes to light very quickly in the form of the programmed stop mentioned earlier.

10. Test Problems Done by This Method. The differential equation problems used to develop the program and to rectify programming errors were those for the sine function and the exponential function. The normal truncation error for these "well-behaved" problems was found to agree with (18).

A test problem to exercise the automatic variable interval feature thoroughly and to verify the behavior for discontinuous f was then devised as follows:

$$\frac{dy}{dx} = \begin{cases} 0 & \text{for } |x - \tfrac{1}{2}| \geqq 2^{-31} \\ 2^5 & \text{for } |x - \tfrac{1}{2}| < 2^{-31} \end{cases} \tag{25}$$

to be integrated from 0 to 1 with h_0 specified as 2^{-8} and β^{-4} specified as 2^{-34}. This involves having the computer search the x-axis efficiently for an extremely narrow region in which $f \neq 0$, finding the area under the curve in this narrow region very

TABLE 2

Steps	h	x	$y \cdot 2^{20}$	"*Correct*" $y \cdot 2^{20}$
0	2^{-8}	0	0	0
157	2^{-38}	$1/2 - 2^{-31}$	0	0
169	2^{-32}	$1/2$	.015 564	.015 564
176	2^{-38}	$1/2 + 2^{-31}$	.031 149	.031 128
370	2^{-8}	1	.031 128	.031 128

TABLE 3

Steps	h	x	$y \cdot 2^{20}$	*Error*
0	2^{-8}	$-1/2$	0	
202	2^{-31}	-2^{-30}	.098 177	.000 002
214	2^{-36}	0	.196 352	.000 003
227	2^{-32}	2^{-30}	.294 527	.000 003
505	2^{-8}	$1/2$	.392 700	.000 001

accurately and then searching the rest of the x-interval at high speed again. The performance on problem (25) is shown in Table 2.

The "correct" y means the exact area of the figure obtained by joining the consecutive pair of points sampled, with $h = 2^{-38}$, at which f changes, by a straight line. The interval actually increased 64-fold temporarily between corner and center of the curve. The somewhat slower recovery of the interval on the increasing-interval side is exhibited in the difference between 194 steps from $x = 1/2 + 2^{-31}$ to $x = 1$ and 157 steps from $x = 0$ to $x = 1/2 - 2^{-31}$. The recovery of the interval to $h_0 = 2^{-8}$ *at all* is evidence that roundoff noise does not persist in the "remembered" quantities.

A test problem similar to the above with a very narrow but smooth analytic curve was also treated:

$$\frac{dy}{dx} = 2^7 \frac{(2^{-30})^2}{x^2 + (2^{-30})^2} \tag{26}$$

to be integrated from $x = -1/2$ to $x = +1/2$ with h_0 specified as 2^{-8} and β^{-e} specified as 2^{-32}. The result of this computation is given in Table 3.

The interval evidently did not have to decrease so much in this case because of the smoother curve to be integrated. The same comments in regard to increasing h apply here as in the previous example. The accumulated error is much less than 2^{-32} because of the simple symmetrical character of the curve being integrated.

Next, a typical unstable differential equation was treated:

$$\frac{dy}{dx} = \frac{20y}{x}; \qquad y = \tfrac{1}{2} \quad \text{at} \quad x_0 = \tfrac{1}{2} \tag{27}$$

to be integrated from $x = 1/2$ to $x = 1$ with $h_0 = 2^{-4}$ and $\beta^{-e} = 2^{-25}$. Results are given in Table 4.

This illustrates the quality of the starting process in keeping the early truncation error small, a very important consideration in this case because such early errors are ultimately magnified one millionfold. Six significant decimals are preserved correct through 63 steps, in each of which the solution increases by 25 per cent on the average. The final error exceeds 2^{-25} because significance cannot increase.

Each of the above tests required only 3 to 15 seconds of computer time, and some sort of longer test seemed appropriate. As such the author chose Bessel's differential equation of order 16, and in particular, to find $J_{16}(z)$ by integrating from $z = 6$ to $z \cong 6000$. In this range the function begins very small, increases monotonically and rapidly over 200,000-fold, and then makes almost 1000 complete oscillations. We put $z = 2^{12}x$ and $h_0 = 2^{-13}$ and $\beta^{-e} = 2^{-23}$ and 2^{-28} respectively, for two tests. Tables 5 and 6 show the results of this computation.

TABLE 4

Steps	x	y	*Error*
0	.50	.000 000 476 837	
1	.507 812 5	.000 000 650 187	$.0^{11}1$
63	1.0	.500 000 546 694	.000 000 55

TABLE 5
($\beta^{-e} = 2^{-23}$)

Steps	z	$J_{16}(z)$	*Error*	$J'_{16}(z)$	*Error*
0	6.0	$.0^{5}1\ 201\ 950$		$.0^{5}2\ 986\ 480$	
1	6.125	$.0^{5}1\ 633\ 713$	$.0^{11}1$	$.0^{5}3\ 963\ 765$	$.0^{11}1$
77	16.0	.177 453 370	$.0^{5}0\ 177$	.062 487 955	$.0^{5}0\ 066$
93	18.0	.261 082 210	$.0^{5}0\ 266$	.003 519 524	$.0^{5}0\ 005$
109	20.0	.145 179 990	$.0^{5}0\ 150$	−.116 956 059	$-.0^{5}0\ 118$
49 005	6132.0	.004 126 972	$-.0^{5}3\ 498$	.009 311 583	
49 021	6134.0	.006 748 858	$-.0^{5}0\ 809$	−.007 627 018	
49 037	6136.0	−.009 741 657	$.0^{5}4\ 174$	−.002 961 758	
49 053	6138.0	.001 359 819	$-.0^{5}2\ 666$	.010 089 144	

TABLE 6
($\beta^{-e} = 2^{-23}$)

Steps	z	$J_{16}(z)$	*Error*	$J'_{16}(z)$	*Error*
0	6.0	$.0^{5}1\ 201\ 950$		$.0^{5}2\ 986\ 480$	
1	6.125	$.0^{5}1\ 633\ 713$	$.0^{11}1$	$.0^{5}3\ 963\ 765$	$.0^{11}1$
99	16.0	.177 453 297	$.0^{5}0\ 104$	.062 487 925	$.0^{5}0\ 036$
126	18.0	.261 082 096	$.0^{5}0\ 152$	.003 519 520	$.0^{5}0\ 001$
156	20.0	.145 179 923	$.0^{5}0\ 083$	−.116 956 010	$-.0^{5}0\ 069$
98 709	6132.0	.004 130 418	$-.0^{5}0\ 052$	.009 314 069	
98 741	6134.0	.006 749 685	$.0^{5}0\ 018$	−.007 631 186	
98 773	6136.0	−.009 745 792	$.0^{5}0\ 039$	−.002 960 774	
98 805	6138.0	.001 362 434	$-.0^{5}0\ 051$	.010 092 495	

Some of the properties of the automatic interval control are well illustrated by these two tables. In spite of our asking for less than full register accuracy, the computer starts accurately enough and with a small enough interval in both cases so that the initial truncation error is half the least count, for it recognizes via test (22a) that early errors may be magnified by the instability of the differential equation itself. The ultimate error is somewhat but not much larger than asked for, as it must be expected to be because of significance considerations. The interval is halved over most of the range and the error drops by just about 2^{-6} as between Table 5 and Table 6 (due allowance being made for the change of phase of the error between the two calculations). In the calculation of Table 6 we end up with almost as many correct significant figures as were given initially. A further increase in e (and in computing time) would presumably improve the preservation of significance a little more.

We emphasize that the above treatment of the Bessel equation is not claimed to be a good way of calculating Bessel functions, but was chosen purposely to illustrate how the method handles a rather "ill-behaved" problem.

Appendix A. The working formulas and truncation errors for degrees $m = 2$ through 6 are collected here.

$m = 2$

$$
\begin{aligned}
y(x+h) &= y(x) + h\{f(x) + a(x) + \tfrac{5}{12}[f(x+h) - f^p]\} \\
f^p &= f(x) + 2a(x) \qquad \text{(A1)} \\
a(x+h) &= a(x) + \tfrac{1}{2}[f(x+h) - f^p]
\end{aligned}
$$

$$E_t = 1 \cdot \frac{h^4}{4!} y^{IV}$$

$m = 3$

$$
\begin{aligned}
y(x+h) &= y(x) + h\{f(x) + a(x) + b(x) + \tfrac{3}{8}[f(x+h) - f^p]\} \\
f^p &= (x) + 2a(x) + 3b(x) \qquad \text{(A2)} \\
a(x+h) &= a(x) + 3b(x) + \tfrac{3}{4}[f(x+h) - f^p] \\
b(x+h) &= b(x) + \tfrac{1}{4}[f(x+h) - f^p]
\end{aligned}
$$

$$E_t = (25/6)\frac{h^5}{5!} y^V$$

$m = 4$

$$
\begin{aligned}
y(x+h) &= y(x) + h\{f(x) + a(x) + b(x) + c(x) + \tfrac{251}{720}[f(x+h) - f^p]\} \\
f^p &= f(x) + 2a(x) + 3b(x) + 4c(x) \qquad \text{(A3)} \\
a(x+h) &= a(x) + 3b(x) + 6c(x) + \tfrac{11}{12}[f(x+h) - f^p] \\
b(x+h) &= b(x) + 4c(x) + \tfrac{1}{2}[f(x+h) - f^p] \\
c(x+h) &= c(x) + \tfrac{1}{24}[f(x+h) - f^p]
\end{aligned}
$$

$$E_t = (27/2)\frac{h^6}{6!} y^{VI}$$

$m = 5$

$$
\begin{aligned}
y(x+h) &= y(x) + h\{f(x) + a(x) + b(x) + c(x) + d(x) + \tfrac{95}{288}[f(x+h) - f^p]\} \\
f^p &= f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x) \qquad \text{(A4)} \\
a(x+h) &= a(x) + 3b(x) + 6c(x) + 10d(x) + \tfrac{25}{24}[f(x+h) - f^p] \\
b(x+h) &= b(x) + 4c(x) + 10d(x) + \tfrac{35}{24}[f(x+h) - f^p] \\
c(x+h) &= c(x) + 5d(x) + \tfrac{5}{16}[f(x+h) - f^p] \\
d(x+h) &= d(x) + \tfrac{1}{120}[f(x+h) - f^p]
\end{aligned}
$$

$$E_t = (863/12)\frac{h^7}{7!} y^{VII}$$

$m = 6$

$$
\begin{aligned}
y(x+h) &= y(x) \\
&\quad + h\{f(x) + a(x) + b(x) + c(x) + d(x) + e(x) \\
&\qquad + \tfrac{19087}{60480}[f(x+h) - f^p]\} \\
f^p &= f(x) + 2a(x) + 3b(x) + 4c(x) + 5d(x) + 6e(x) \\
a(x+h) &= a(x) + 3b(x) + 6c(x) + 10d(x) + 15e(x) \\
&\qquad + \tfrac{1375}{1344}[f(x+h) - f^p] \qquad \text{(A5)} \\
b(x+h) &= b(x) + 4c(x) + 10d(x) + 20e(x) \\
&\qquad + \tfrac{35}{24}[f(x+h) - f^p] \\
c(x+h) &= c(x) + 5d(x) + 15e(x) \\
&\qquad + \tfrac{11}{24}[f(x+h) - f^p] \\
d(x+h) &= d(x) + 6e(x) \\
&\qquad + \tfrac{1}{20}[f(x+h) - f^p]
\end{aligned}
$$

$$e(x+h) = e(x) + \tfrac{1}{720}\,[f(x+h) - f^{\circ}]$$

$$E_t = 513\,\frac{h^8}{8!}\,y^{VIII}$$

Appendix B. The flow chart (Figure 3) presented here is probably in terms which are general enough to apply to most stored-program computers. As shown, it pro-

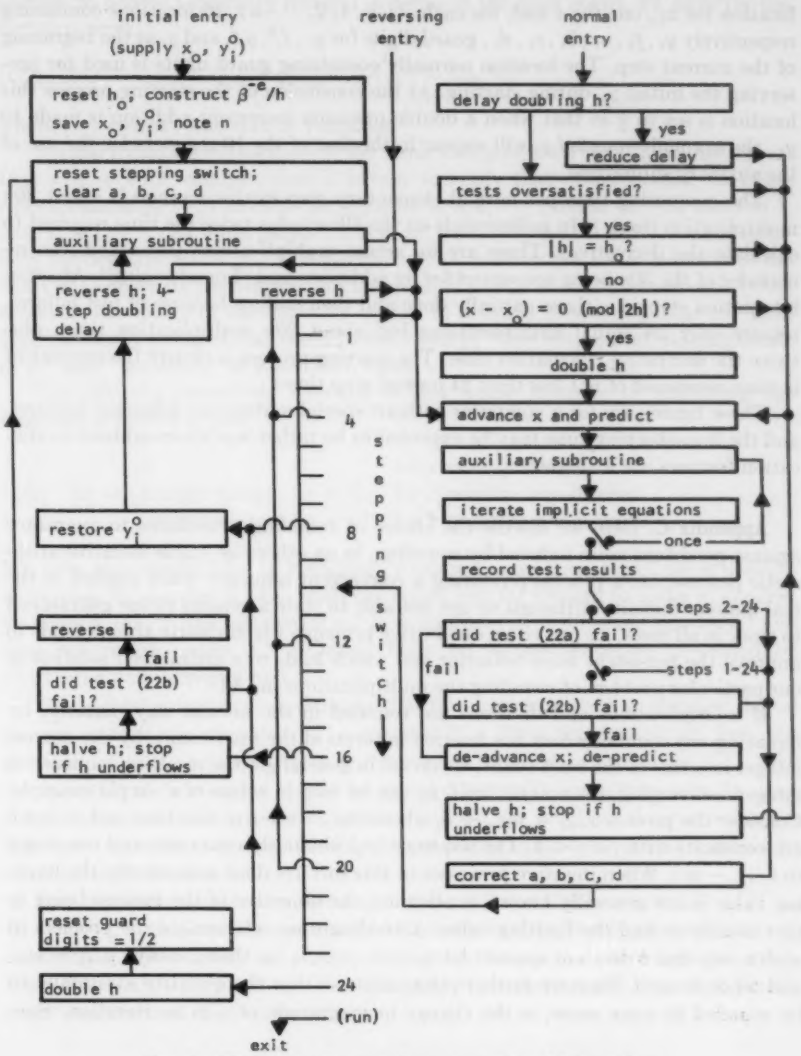

FIG. 3.—Flow Chart for one elementary step of integration.

vides for one elementary step of integration per entry into the routine, so that a master program can supervise the general course of the computation with complete flexibility. It also appeals to an "auxiliary subroutine" (closed) to calculate $f(x, y)$ given x and y, for complete flexibility as to what system of differential equations is being treated. The parameters which must be supplied are: the order n, the location of the auxiliary subroutine h_0, the accuracy parameter e, and the location of a working storage of $2 + 10n$ memory locations. The working storage contains a location for x_0, one for x and, for each $i (i = 1, 2, \cdots n)$, 10 locations containing respectively $y_i, f_i, a_i, b_i, c_i, d_i$, guard digits for y_i, f_i^p, y_i^p, and y_i at the beginning of the current step. The location normally containing guard digits is used for preserving the initial y_i^0 during starting. At the conclusion of the starting process this location is set to $\frac{1}{2}$ so that when a double precision increment-addition is made to y_i, the normally *rounded* y_i will appear in the first of the 10 registers for the use of the auxiliary subroutine.

The computing time per normal elementary step in this method is about $30n$ multiplication times ($21n$ milliseconds on the Illiac) plus twice the time required to calculate the derivatives. There are $6n$ actual multiplications performed, the remainder of the $30n$ being accounted for by additions and "housekeeping". Abortive integration steps, i.e. those partially done and then undone because of test failures, require only $2n$ actual multiplications but about $20n$ multiplication times plus twice the derivative calculation time. The starting process is clearly the equivalent in time consumed of not less than 24 normal step times.

These figures are for a computer without special address modification features, and the housekeeping time may be expected to be rather less where address modification features are available.

Appendix C. Here we discuss the choice of rounding procedures to guarantee against persistent noise induced by rounding, in an otherwise stable iterative arithmetic process, i.e. a process producing a convergent sequence when applied in the real number domain. Although we are not able to state a general recipe guaranteed to work in all cases, we can cite a qualitative principle which clearly always tends to improve the persistent noise behavior and which leads to a guaranteed solution in our particular problem of rounding the multiplications in (A4).

If multiplications and divisions are rounded in the normal way, namely, by replacing any number which is a fraction in terms of the least count, by the nearest integer in terms of the least count, we do *not* in general get the resulting sequence of integers converging to a unique limit, as can be seen in terms of a simple example. Consider the process $x_{n+1} = mx_n + b$, where the x's are real numbers and m and b are constants with $| m | < 1$. The sequence $\{x_n\}$ obviously converges and converges to $b/(1 - m)$. When iterative processes of this sort are done numerically the limiting value is not generally known in advance, the objective of the process being in fact usually to find the limiting value. Accordingly we reformulate the problem in such a way that b does not appear: let $y_n = x_n - x_{n-1}$, so that y_n obeys $y_{n+1} = my_n$ and tends to zero. Then we further reformulate so that the quantity eventually to be rounded in some sense, is the change in magnitude of y in an iteration. Spe-

cifically, we write

$$(C1) \qquad y_{n+1} = \pm y_n \mp (1 - |m|)y_n = \pm y_n \mp \mu y_n$$

according as $m > 0$ or $m < 0$. Observe that $0 < \mu < 1$.

The digital (integer) process corresponding to the real number process (C1) involves rounding the product μy_n to an integer according to some rule. Using an asterisk to denote a quantity integral in terms of the least count, we have for the digital process:

$$(C2) \qquad y^*_{n+1} = \pm(y_n^* - [\mu y_n^*])$$

where [ ] means some sort of rounding.

Normal rounding causes most of the sequences generated by (C2) to misbehave. If $\mu = \frac{1}{4} - \epsilon$ for example, then it is easy to verify that under normal rounding rules an initial $y_0^* = 0$ leads to the sequence $0, 0, 0, \cdots$; initial $y_0^* = 1$ leads to $1, \pm 1, 1, \pm 1, \cdots$; all other positive initial y_0^* lead to $2, \pm 2, 2, \pm 2 \cdots$; and similarly for negative initial y^*. This general sort of misbehavior is not peculiar to the value of μ chosen for illustration, but is typical of most μ's. In the formulation (C2), however, the source of the difficulty is easy to discern: it is merely that the term $[\mu y_n^*]$ normally rounded may often vanish when y_n^* does not, so that the magnitude of y_n^* may "get stuck" at a non-zero value.

The difficulty is entirely removed in this simple example by redefining the rounding process so that

$$(C3) \qquad [x] \equiv \begin{cases} x & \text{for } x \text{ exactly integral} \\ \text{integer nearest } (x + \frac{1}{2}) & \text{for } x \text{ positive non-integral} \\ \text{integer nearest } (x - \frac{1}{2}) & \text{for } x \text{ negative non-integral} \end{cases}$$

We term this special kind of rounding "rounding away from zero," for it consists of moving the number x *away* from the origin just far enough to make it integral. So defined, $[\mu y_n^*]$ does not exceed y_n^* in magnitude, is of the same sign as y_n^* and does not vanish unless y_n^* vanishes. Thus, all integer sequences generated by (C2) must now converge to 0.

The general principle is accordingly that if we can formulate an iterative digital process so that the quantity to be rounded is a correction subtracted from the previous value of an integer variable intended to converge to zero, as in (C2), then the quantity to be rounded should be rounded generally away from zero. In more complicated cases where several integer variables are involved the correction (in the above sense) to each may be a function of all the variables; but still it should be rounded away from zero.

Our particular problem consists of rounding the multiplications $A[\ ]$, $B[\ ]$, $C[\ ]$, $D[\ ]$ in the working equations (A4). Suppose that f tends asymptotically to a constant and consider what may happen when $a \cdots d$ have become small. Then $f(x + h) - f(x)$ will cancel out of (A4) at some stage, and thereafter the relevant equations of the process will be:

$$
\begin{aligned}
a^*_{n+1} &= a_n^* + 3b_n^* + 6c_n^* + 10d_n^* \\
&\quad - [(25/24)(2a_n^* + 3b_n^* + 4c_n^* + 5d_n^*)] \\
b^*_{n+1} &= b_n^* + 4c_n^* + 10d_n^* \\
&\quad - [\ 35/72\ (2a_n^* + 3b_n^* + 4c_n^* + 5d_n^*)] \\
c^*_{n+1} &= c_n^* + 5d_n^* \\
&\quad - [\ 5/48\ \ (2a_n^* + 3b_n^* + 4c_n^* + 5d_n^*)] \\
d^*_{n+1} &= d_n^* \\
&\quad - [\ 1/120\ (2a_n^* + 3b_n^* + 4c_n^* + 5d_n^*)]
\end{aligned}
\qquad \text{(C4)}
$$

where the asterisk signifies a quantity integral in terms of the least count, and the [ ] symbolizes rounding. Note that these equations are in just the form we require to apply the "rounding away from zero" principle, since the terms $3b_n^*$, $4c_n^*$ etc. are integral and have no effect on the behavior of the rounding.

Normal rounding in equations (C4) leads to persistent roundoff noise. The rounding process is so non-linear that we have no analytical theory and must work out specific numerical examples. Two examples of indefinitely persisting (cyclic) roundoff noise are:

n	a^*	b^*	c^*	d^*	n	a^*	b^*	c^*	d^*
0	1	0	0	0	0	0	1	0	1
1	−1	−1	0	0	1	5	7	4	1
2	1	1	1	0	2	6	8	4	1
3	1	1	0	0	3	5	6	3	1
4	−1	−1	−1	0	4	4	6	3	1
5	−1	−1	0	0	5	5	7	4	1
6	1	1	1	0	6	6	8	4	1
		etc.					etc.		

As we saw in Section 8, any behavior like this (and there are many cases of it) can frustrate the interval control in its attempts to increase the interval when the interval obviously ought to be increased. Curiously enough, the persistent cycles of roundoff noise contribute practically no error to y, for the contribution to y, averaged over a repetitive noise cycle, is no more than about $h/60$ times the least count. However, proper behavior of the interval control alone is enough reason for rectifying the roundoff behavior.

The simplest change in rounding which suggests itself is rounding all four multiplications in (C4) away from zero. However, such a simple remedy does not work, for it represents too drastic a modification of the fourth equation of (C4). It implies in fact that d^* must change unless $(2a^* + 3b^* + 4c^* + 5d^*)$ is zero, and persistent oscillation of d^* results inevitably. After some experimentation the author has concluded that the best rule is: round the first three multiplications in (C4) away from zero according to (C3), but for the fourth multiplication move the multiplicand $(2a^* + 3b^* + 4c^* + 5d^*)$ away from zero by 16 units and then multiply by $\frac{1}{1\,8\,0}$, rounding *normally*. The treatment of the fourth multiplication is a "partial" rounding away from zero or a less drastic modification of normal rounding, but

clearly in the same spirit. The rounding rules thus finally fixed upon will cause every initial quadruple of integers to converge to (0, 0, 0, 0), as was verified by letting the computer treat every case. Actually, all initial quadruples of integers between −2 and 2 inclusive were examined, and all tend to (0, 0, 0, 0). The average number of steps to arrive at (0, 0, 0, 0) is $\frac{1}{3}^3$ and the maximum is 14. If we move the multiplicand of the last multiplication only 12 units instead of 16, one persistent cycle appears. If we move it 14, 16, respectively 18 units all quadruples converge to (0, 0, 0, 0) but the average number of steps to clear begins to increase. Thus 16 seems a safe compromise.

These principles may be of help in deciding how to round the arithmetic in other iterative digital processes, such as solving systems of implicit equations. In our present state of knowledge of the subject a certain amount of experimenting of the sort described above will probably have to be done in every individual case more complicated than the one-variable case. The general reason for stabilizing roundoff noise in these ways is to improve the functioning of tests-for-end, for such tests are subject to the same difficulties as test (22b) in our procedure.

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Triple Product Integrals of Laguerre Functions

By J. Gillis and M. Shimshoni

1. Introduction. We shall use the following standard definitions for Laguerre polynomials (1) and Laguerre functions (2):

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \binom{n}{r} \frac{x^r}{r!} \tag{1}$$

$$\lambda_n(x) = e^{-x/2} L_n(x) \tag{2}$$

The Laguerre functions are known to constitute a complete orthonormal set in $L^2(0, \infty)$. Given a differential equation over $0 \leqq x < \infty$ one naturally thinks, therefore, of the possibility of solution by expansion as a series of Laguerre functions. However, for this to be useful for non-linear differential equations, we need to be able to expand the product of two Laguerre functions as a linear series of these functions. The main part of this paper is devoted to methods for effecting the expansion, and we shall also give an application of the results. The similar problem for Laguerre polynomials has been solved by Watson [5], and methods for computing the expansion coefficients discussed by Gillis and Weiss [3].

We may write

$$\lambda_r \lambda_s = \sum_{t=0}^{\infty} C_{rst} \lambda_t \tag{3}$$

where

$$C_{rst} = \int_0^\infty \lambda_r(x) \lambda_s(x) \lambda_t(x)\, dx. \tag{4}$$

We shall also write this as $C_{r,s,t}$.

It follows that the coefficients will be symmetric in all three suffixes. We give below a table of these coefficients for $0 \leqq r \leqq s \leqq t \leqq 10$, and also expressions of C_{rst} as polynomials in t for $0 \leqq r \leqq s \leqq 3$.

In Section 2 we discuss a number of formulas for C_{rst}. Those in (a) and (b) involve three-fold summations and apply to general r, s, t. In (c) we obtain a simple sum formula valid for the case $r = 0$ and, in (d), a double sum for the case $r = s$. In Section 3 we shall derive two recurrence formulas for the coefficients C_{rst}. As a check on the stability of these latter formulas in practice, it is advisable to have comparatively simple alternative methods for computing C_{rst} for certain particular triads r, s, t. For this purpose the formulas of Section 2 (c), (d) can be of use.

2. Explicit Formulas.

(a)

$$\begin{aligned} C_{rst} &= \int_0^\infty e^{-3x/2} \sum_{\alpha=0}^{r} (-1)^\alpha \binom{r}{\alpha} \frac{x^\alpha}{\alpha!} \sum_{\beta=0}^{s} (-1)^\beta \binom{s}{\beta} \frac{x^\beta}{\beta!} \sum_{\gamma=0}^{t} (-1)^\gamma \binom{t}{\gamma} \frac{x^\gamma}{\gamma!}\, dx \\ &= \frac{2}{3} \sum_{\alpha,\beta,\gamma} \left(-\frac{2}{3}\right)^{\alpha+\beta+\gamma} \binom{r}{\alpha} \binom{s}{\beta} \binom{t}{\gamma} \frac{(\alpha+\beta+\gamma)!}{\alpha!\beta!\gamma!} \end{aligned} \tag{5}$$

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(b) Since

$$x^n = n! \sum_{\alpha=0}^{n} (-1)^\alpha \binom{n}{\alpha} L_\alpha(x), \qquad \text{(cf. [1])},$$

it follows that

$$\begin{aligned} L_r(x)L_s(x) &= \sum_{p=0}^{r} (-1)^p \binom{r}{p} \frac{x^p}{p!} \sum_{q=0}^{s} (-1)^q \binom{s}{q} \frac{x^q}{q!} \\ &= \sum_{p=0}^{r} \sum_{q=0}^{s} (-1)^{p+q} \binom{r}{p} \binom{s}{q} \binom{p+q}{p} \sum_{t=0}^{p+q} (-1)^t \binom{p+q}{t} L_t(x), \end{aligned} \tag{6}$$

i.e.,

$$L_r(x)L_s(x) = \sum_t A_{rst} L_t(x) \tag{7}$$

where

$$A_{rst} = \sum_{p=0}^{r} \sum_{q=0}^{s} (-1)^{p+q+t} \binom{r}{p} \binom{s}{q} \binom{p+q}{p} \binom{p+q}{t}. \tag{8}$$

Now [1],

$$e^{-x/2} = 2 \sum_{p=0}^{\infty} 3^{-p-1} L_p(x) \tag{9}$$

and so

$$\begin{aligned} \sum_t C_{rst} L_t(x) = e^{-x/2} L_r(x) L_s(x) &= 2 \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} 3^{-p-1} A_{rsm} L_p(x) L_m(x) \\ &= 2 \sum_{p=0}^{\infty} \sum_{m=0}^{\infty} \sum_{t=|m-p|}^{m+p} 3^{-p-1} A_{rsm} A_{mpt} L_t(x). \end{aligned} \tag{10}$$

Hence

$$\begin{aligned} C_{rst} = 2 \sum_{p,m} 3^{-p-1} A_{rsm} A_{mpt} = 2 \sum_{m,p,\alpha,\beta,\gamma,\delta} (-1)^{\alpha+\beta+\gamma+\delta+m+t} 3^{-p-1} \\ \cdot \binom{r}{\alpha} \binom{s}{\beta} \binom{\alpha+\beta}{\alpha} \binom{\alpha+\beta}{m} \binom{m}{\gamma} \binom{p}{\delta} \binom{\gamma+\delta}{\gamma} \binom{\gamma+\delta}{t} \end{aligned} \tag{11}$$

The sum over p yields

$$\sum_{p \geq \delta} \binom{p}{\delta} 3^{-p-1} = 2^{-\delta-1}, \tag{12}$$

while the sum over m is

$$\begin{aligned} \sum_m (-1)^m \binom{\alpha+\beta}{m} \binom{m}{\gamma} = (-1)^\gamma \binom{\alpha+\beta}{\gamma} \sum_m (-1)^{m-\gamma} \binom{\alpha+\beta-\gamma}{m-\gamma} \\ = 0 \qquad \text{if} \quad \gamma \neq \alpha+\beta, \\ = (-1)^\gamma \quad \text{if} \quad \gamma = \alpha+\beta. \end{aligned} \tag{13}$$

Hence

$$C_{rst} = \sum_{\alpha=0}^{r} \sum_{\beta=0}^{s} \sum_{\delta=0}^{\infty} \frac{(-1)^{\alpha+\beta+t+\delta}}{2^\delta} \binom{r}{\alpha} \binom{s}{\beta} \binom{\alpha+\beta}{\alpha} \binom{\alpha+\beta+\delta}{\delta} \binom{\alpha+\beta+\delta}{t} \tag{14}$$

(c) Take Laplace transforms of both sides of (3). This gives [2]

$$\binom{r+s}{r} p^{r+s}(p-1)^{-r-s-1}\,{}_2F_1[-r,-s;-r-s;1-p^{-2}] = \sum_t C_{rst}(p-\tfrac{1}{2})^t(p+\tfrac{1}{2})^{-t-1}. \tag{15}$$

Writing $q = (p-\frac{1}{2})/(p+\frac{1}{2})$, we obtain

$$\sum_t C_{rst}\,q^t = 2\binom{r+s}{r}(1+q)^{r+s}(3-q)^{-r-s-1} \cdot {}_2F_1\left[-r,-s;-r-s;\frac{(3-q)(3q-1)}{(q+1)^2}\right]. \tag{16}$$

In the special case $r = 0$ this leads to the simple result that

$$C_{0st} = \text{coefficient of } q^t \text{ in } 2(1+q)^s(3-q)^{-s-1} = 2\sum_{n=0}^{t} 3^{-n-s-1}(s+n)!\{n!\,(t-n)!\,(s+n-t)!\}^{-1} \tag{17}$$

We could similarly use (16) to obtain expressions for C_{rst} for small nonzero values of r, but the formulas soon become prohibitively complicated. It follows incidentally from (17) that $C_{0st} > 0$ for all s, t and that $3^{s+t+1}C_{0st}$ is an even integer.

(d) From the generating function [1]

$$\sum_{n=0}^{\infty} L_n(z)u^n = (1-u)^{-1}\exp\{uz(u-1)^{-1}\} \tag{18}$$

we obtain

$$\sum_{n=0}^{\infty} \lambda_n(z)u^n = (1-u)^{-1}\exp\{\tfrac{1}{2}z(u+1)(u-1)^{-1}\} \tag{19}$$

Hence

$$\lambda_r(z)\lambda_s(z) = (2\pi i)^{-2}\int^{(0+)}\int^{(0+)} (1-u)^{-1}(1-v)^{-1}u^{-r-1}v^{-s-1} \cdot \exp\left\{\frac{1}{2}z\left[\frac{u+1}{u-1}+\frac{v+1}{v-1}\right]\right\} du\,dv. \tag{20}$$

Choose w so that

$$(w+1)/(w-1) = (u+1)/(u-1) + (v+1)/(v-1) \tag{21}$$

and write

$$\exp\{\tfrac{1}{2}(w+1)(w-1)^{-1}\} = (1-w)\sum_{t=0}^{\infty}\lambda_t(z)w^t.$$

Then

$$\lambda_r(z)\lambda_s(z) = \sum_{t=0}^{\infty}(2\pi i)^{-2}\lambda_t(z)\int^{(0+)}\int^{0(+)} F_t(u,v)u^{-r-1}v^{-s-1}\,du\,dv \tag{22}$$

where

$$F_t(u, v) = 2(1 + u + v - 3uv)^t(3 - u - v - uv)^{-t-1}. \tag{23}$$

It follows that C_{rst} is the coefficient of $u^r v^s$ in $F_t(u, v)$.

By putting $u = 0$ we can calculate C_{0st} and immediately recover (17). Formulas for other small values of r can be obtained by expanding $\left[\frac{\partial^r}{\partial u^r} F_t(u, v)\right]_{u=0}$ in powers of v. However, there is another case in which this can be used to obtain a closed expression for C_{rst}, namely, when $r = s$. For this purpose we write

$$F_t(u, v) = (-1)^t 2^{-2t-1}(4\alpha - \beta)^t(1 - \tfrac{1}{4}\beta)^{-t-1} \tag{24}$$

where

$$\alpha = uv$$

and

$$\beta = (u + 1)(v + 1). \tag{25}$$

We note that the coefficient of $u^r v^r$ in the expansion of β^n is $\binom{n}{r}^2$. But, by (24),

$$F_t(u, v) = (-1)^t 2^{-2t-1} \sum_{m=0}^{t} \sum_{n=0}^{\infty} (-1)^m \binom{t}{m} (4\alpha)^{t-m} \beta^{m+n} 4^{-n} \binom{t+n}{n}. \tag{26}$$

It follows that C_{rrt}, the coefficient of $u^r v^r$ in the expression $F_t(u, v)$, is

$$\tfrac{1}{2}(-1)^t \sum_{m=0}^{t} \sum_{n=0}^{\infty} (-1)^m 2^{-2(m+n)} \binom{t}{m} \binom{t+n}{n} \binom{m+n}{r-t+m}^2. \tag{27}$$

The table of numerical values of C_{rst} suggests the conjecture that C_{rrt} has the sign of $(-1)^t$ for $0 \leqq t \leqq r$. This would mean that the double sum in (27) is always positive for $0 \leqq t \leqq r$, but we have not been able to prove the conjecture.

3. Recurrence Relations.

(a) *A Two-Index Recurrence Relation.* From the relation

$$x\lambda_r(x) = -(r + 1)\lambda_{r+1}(x) + (2r + 1)\lambda_r(x) - r\lambda_{r-1}(x), \tag{28}$$

[1], we deduce that

$$\int_0^\infty x\lambda_r(x)\lambda_{r+1}(x)\, dx = -(r + 1) \tag{29}$$

and

$$\int_0^\infty x\lambda_r^2\, dx = 2r + 1, \tag{30}$$

while

$$\int_0^\infty x\lambda_r\lambda_s\, dx = 0 \qquad \text{if } s \neq r, r \pm 1. \tag{31}$$

Multiplying both sides of (28) by λ_s and using (3), we get

$$\sum_{t=0}^{\infty} C_{rst}x\lambda_t = -(r+1)\sum_{t=0}^{\infty} C_{r+1,s,t}\lambda_t + (2r+1)\sum_{t=0}^{\infty} C_{rst}\lambda_t - r\sum_{t=0}^{\infty} C_{r-1,s,t}\lambda_t . \tag{32}$$

We now multiply both sides of (32) by λ_t and integrate from 0 to ∞, using (29), (30), (31), and immediately obtain

$$(r+1)C_{r+1,s,t} = tC_{r,s,t-1} + 2(r-t)C_{r,s,t} + (t+1)C_{r,s,t+1} - rC_{r-1,s,t} . \tag{33}$$

This is the required recurrence relation. It can be used quite effectively, in conjunction with (17), to compute a table of C_{rst}. We first compute C_{0st} by (17) for an adequate range of (s, t). It then follows from (33) that

$$C_{1st} = tC_{0,s,t-1} - 2tC_{0st} + (t+1)C_{0,s,t+1} \tag{34}$$

and this gives us C_{1st}, etc. The computation is reasonably stable, although, as a safety precaution, one would carry more digits than are actually needed. Actually, the explicit formula for C_{1st}, obtainable from (17) and (34), is

$$C_{1st} = 2\sum_{n=0}^{t} 3^{-s-t-2}(s-1+n)!\{n!(t-n)!\,(s-t+n+1)!\}^{-1}F_{nst} \tag{35}$$

where

$$F_{nst} = (4s+1)n^2 + (8s^2 - 12st + 6s + 1)n + s[4s^2 - (12t-5)s + 9t(t-1) + 1]. \tag{36}$$

A similar, but much more complicated formula can be derived from this for C_{2st}. The corresponding formulas for C_{rst} become very complicated for larger values of r.

For work with an electronic computer it would be better to have a method for generating the C_{rst} as required rather than to store a table of these coefficients. In the next paragraph we propose a method of achieving this by a recurrence relation which operates on only one of the indices.

(b) *An Alternative Recurrence Relation.* We recall that λ_r satisfies the differential equation

$$x\lambda_r'' + \lambda_r' + (r + \tfrac{1}{2} - \tfrac{1}{4}x)\lambda_r = 0. \tag{37}$$

Hence, if we write

$$u_r = x^{1/2}\lambda_r , \tag{38}$$

we can easily verify that

$$u_r'' + P_r u_r = 0, \tag{39}$$

where

$$P_s = \frac{1}{4x^2} + \frac{r+\frac{1}{2}}{x} - \frac{1}{4}. \tag{40}$$

It may then be deduced, by a standard procedure, that $y = u_r u_s$ satisfies the differential equation

$$\begin{aligned} x^4 y^{(IV)} + x^3 y''' + x^2(1 + 2\sigma x - x^2)y'' \\ - x(2 + \sigma x + x^2)y' + (2 + \sigma x + \delta^2 x^2)y = 0 \end{aligned} \tag{41}$$

where

$$\left.\begin{aligned} \sigma &= r + s + 1 \\ \delta &= |r - s| \end{aligned}\right\}. \tag{42}$$

Let $z = \lambda_r \lambda_s$. Then, by (38),

$$y = xz \tag{43}$$

and this may be substituted directly into (41), giving

$$\mathfrak{L}(z) = x^2 z^{(IV)} + 5xz''' + (4 + 2\sigma x - x^2)z'' + 3(\sigma - x)z' + (\delta^2 - 1)z = 0. \tag{44}$$

We now write

$$z = \sum_t C_{rst} \lambda_t \tag{45}$$

and substitute in $\mathfrak{L}(z)$.

After a little manipulation, using the properties of Laguerre functions, we obtain

$$\mathfrak{L}(\lambda_t) = \left(\sigma - \rho - \frac{5x}{4}\right)\lambda_t' + \left[\rho^2 - 2\rho\sigma + \delta^2 - \frac{1}{2} + \frac{\sigma + \rho}{2}x - \frac{3}{16}x^2\right]\lambda_t \tag{46}$$

where

$$\rho = t + \tfrac{1}{2}$$

and hence, again going back to basic properties of λ_t,

$$\begin{aligned} x\mathfrak{L}(\lambda_t) = {} & \tfrac{3}{16}(t+1)(t+2)(t+3)\lambda_{t+3} \\ & - \tfrac{1}{16}(t+1)(t+2)(10t - 8\sigma + 13)\lambda_{t+2} \\ & - \tfrac{1}{16}(t+1)[3t^2 + 2t + 2(8\delta^2 + 4\sigma + 1)]\lambda_{t+1} \\ & + [\tfrac{5}{8}(2t+1)(t^2 + t + 1) + \delta^2(t+1) - \sigma(t^2 + t + \tfrac{1}{2})]\lambda_t \\ & - \tfrac{1}{16}t(3t^2 + 4t + 16\delta^2 - 8\sigma + 3)\lambda_{t-1} \\ & - \tfrac{1}{16}t(t-1)(10t - 8\sigma - 3)\lambda_{t-2} + \tfrac{3}{16}t(t-1)(t-2)\lambda_{t-3}. \end{aligned} \tag{47}$$

It follows from (45) and (47) that

$$x\mathfrak{L}(z) = \sum B_t \lambda_t \tag{48}$$

where

$$
\begin{aligned}
B_t = {} & \tfrac{2}{15}(t+3)(t+2)(t+1)C_{r,s,t+3} \\
& - \tfrac{1}{15}(t+2)(t+1)(10t - 8\sigma + 17)C_{r,s,t+2} \\
& - \tfrac{1}{15}(t+1)[3t^2 + 10t + 16\delta^2 - 8\sigma + 10]C_{r,s,t+1} \\
& + [\tfrac{5}{6}(2t+1)(t^2 + t + 1) + \delta^2(2t+1) - \sigma(t^2 + t + \tfrac{1}{2})]C_{rst} \\
& - \tfrac{1}{15}t[3t^2 + 4t + 16\delta^2 + 8\sigma + 3]C_{r,s,t-1} \\
& - \tfrac{1}{15}t(t-1)(10t - 8\sigma - 7)C_{r,s,t-2} \\
& + \tfrac{2}{15}t(t-1)(t-2)C_{r,s,t-3}\,.
\end{aligned} \tag{49}
$$

Since $\mathfrak{L}(z) = 0$ we have, from the completeness of the λ_t's, that

$$
\begin{aligned}
C_{r,s,t+3} = {} & \frac{10t - 8\sigma + 17}{3(t+3)} C_{r,s,t+2} + \frac{3t^2 + 10t + 16\delta^2 - 8\sigma + 10}{3(t+2)(t+3)} C_{r,s,t+1} \\
& - \frac{10(2t^3 + 3t^2 + 3t + 1) + 2\delta^2(2t+1) - 16\sigma(t^2 + t + \tfrac{1}{2})}{3(t+1)(t+2)(t+3)} C_{r,s,t} \\
& + \frac{t(3t^2 - 4t + 16\delta^2 + 8\sigma + 3)}{3(t+1)(t+2)(t+3)} C_{r,s,t-1} \\
& + \frac{t(t-1)(10t - 8\sigma - 7)}{3(t+1)(t+2)(t+3)} C_{r,s,t-2} - \frac{t(t-1)(t-2)}{(t+1)(t+2)(t+3)} C_{r,s,t-3}\,.
\end{aligned} \tag{50}
$$

Suppose, then, we know C_{rst} for $t = 0, 1, 2$. Then, putting $t = 0$ in (50) gives us

$$
\begin{aligned}
18C_{r,s,3} = {} & -2(8\sigma - 17)C_{r,s,2} \\
& + 2(8\delta^2 - 4\sigma + 5)C_{r,s,1} - (2\delta^2 - 8\sigma - 10)C_{r,s,0}
\end{aligned} \tag{51}
$$

and the subsequent terms are all obtainable from the recurrence relation. $C_{r,s,0}$ and $C_{r,s,1}$ can be computed directly from (17) and (35) respectively. It would be possible to develop a corresponding formula for $C_{r,s,2}$ but it would not be very useful. Perhaps the best way to obtain $C_{r,s,2}$ is by use of the relation (33).

It should be remarked that equation (50) is probably the most suitable, among the formulas given above, for use with an electronic computer. For work with a desk computing machine it is rather complicated, and there is little doubt that (33) would prove more useful.

4. Tables. When working by hand, it will generally be convenient to have a table of the numerical values of the C_{rst}. We give this immediately below in Table I for $0 \leqq r \leqq s \leqq t \leqq 10$. Some general formulas for C_{rst} as polynomials in t for $0 \leqq r \leqq s \leqq 3$ are given in Table II.

5. Application. As mentioned in Section 1, our purpose was to apply the above ideas to the solution of non-linear differential equations over a semi-infinite range.

TABLE I

r	s	t	C_{rst}	r	s	t	C_{rst}
0	0	0	.66666667	0	5	8	.07837182
0	0	1	.22222222	0	5	9	.04878490
0	0	2	.07407407	0	5	10	.02839004
0	0	3	.02469136				
0	0	4	.00823045	0	6	6	.16045055
0	0	5	.00274348	0	6	7	.14305131
0	0	6	.00091449	0	6	8	.11236326
0	0	7	.00030483	0	6	9	.07984000
0	0	8	.00010161	0	6	10	.05233831
0	0	9	.00003387				
0	0	10	.00001129	0	7	7	.14885106
				0	7	8	.13475521
0	1	1	.37037037	0	7	9	.10898616
0	1	2	.22222222	0	7	10	.08038815
0	1	3	.10699588				
0	1	4	.04663923	0	8	8	.13945383
0	1	5	.01920439	0	8	9	.12773173
0	1	6	.00762079	0	8	10	.10570201
0	1	7	.00294671				
0	1	8	.00111772	0	9	9	.13163910
0	1	9	.00041773	0	9	10	.12169095
0	1	10	.00015430				
				0	10	10	.12500700
0	2	2	.27160494				
0	2	3	.20027435	1	1	1	−.07407407
0	2	4	.11796982	1	1	2	.17283951
0	2	5	.06127115	1	1	3	.21124829
0	2	6	.02936544	1	1	4	.15089163
0	2	7	.01331098	1	1	5	.08687700
0	2	8	.00579180	1	1	6	.04440380
0	2	9	.00244242	1	1	7	.02103338
0	2	10	.00100482	1	1	8	.00944978
				1	1	9	.00408324
0	3	3	.22405121	1	1	10	.00171233
0	3	4	.18076513				
0	3	5	.12000203	1	2	2	−.04115226
0	3	6	.07021287	1	2	3	.08504801
0	3	7	.03762977	1	2	4	.16369456
0	3	8	.01891085	1	2	5	.15333029
0	3	9	.00904835	1	2	6	.10841843
0	3	10	.00416520	1	2	7	.06553879
				1	2	8	.03580078
0	4	4	.19519382	1	2	9	.01821086
0	4	5	.16532033	1	2	10	.00878492
0	4	6	.11851174				
0	4	7	.07545146	1	3	3	−.02042372
0	4	8	.04399736	1	3	4	.05009399
0	4	9	.02398552	1	3	5	.12508256
0	4	10	.01239969	1	3	6	.14086606
				1	3	7	.11596019
0	5	5	.17527816	1	3	8	.07989974
0	5	6	.15303674	1	3	9	.04897920
0	5	7	.11566665	1	3	10	.02762203

TABLE I—*Continued*

r	s	t	C_{rst}	r	s	t	C_{rst}
1	4	4	−.01188843	2	3	10	.07467760
1	4	5	.03383631				
1	4	6	.09794126	2	4	4	.12284713
1	4	7	.12524062	2	4	5	.04609731
1	4	8	.11552364	2	4	6	−.00911108
1	4	9	.08823376	2	4	7	.01571953
1	4	10	.05941345	2	4	8	.06735022
				2	4	9	.09906079
1	5	5	−.00795949	2	4	10	.10112375
1	5	6	.02489458				
1	5	7	.07889869	2	5	5	.10347339
1	5	8	.11044813	2	5	6	.05127568
1	5	9	.11111857	2	5	7	−.00230192
1	5	10	.09212795	2	5	8	.00330548
				2	5	9	.04431534
1	6	6	−.00582692	2	5	10	.08081952
1	6	7	.01934154				
1	6	8	.06515827				
1	6	9	.09753256	2	6	6	.09128849
1	6	10	.10499757	2	6	7	.05266394
				2	6	8	.00397312
1	7	7	−.00451393	2	6	9	−.00198035
1	7	8	.01560049	2	6	10	.02815229
1	7	9	.05492763				
1	7	10	.08655013	2	7	7	.08275543
				2	7	8	.05253757
1	8	8	−.00363396	2	7	9	.00903873
1	8	9	.01293342	2	7	10	−.00363722
1	8	10	.04709154				
				2	8	8	.07631313
1	9	9	−.00300868	2	8	9	.05177168
1	9	10	.01095104	2	8	10	.01298846
1	10	10	−.00254500	2	9	9	.07120551
				2	9	10	.05073310
2	2	2	.20576132	2	10	10	.06701828
2	2	3	−.00457247				
2	2	4	.00335315	3	3	3	−.05375197
2	2	5	.09053498	3	3	4	.11295704
2	2	6	.13260174	3	3	5	.07694175
2	2	7	.12277939	3	3	6	−.00123814
2	2	8	.09060272	3	3	7	.00106252
2	2	9	.05811997	3	3	8	.05014082
2	2	10	.03385805	3	3	9	.09001406
				3	3	10	.10048996
2	3	3	.15658182				
2	3	4	.03119443	3	4	4	−.03744912
2	3	5	−.01249810	3	4	5	.07177090
2	3	6	04167161	3	4	6	.08787554
2	3	7	.09805416	3	4	7	.02244924
2	3	8	.11549354	3	4	8	−.00731554
2	3	9	.10130123	3	4	9	.01802366

TABLE I—*Continued*

r	s	t	C_{rst}	r	s	t	C_{rst}
3	4	10	.05982871	4	7	10	.02567632
3	5	5	−.02226023	4	8	8	.06531960
3	5	6	.04824409	4	8	9	.02633548
3	5	7	.08353180	4	8	10	−.00693875
3	5	8	.03995357				
3	5	9	−.00156912	4	9	9	.05946498
3	5	10	.00178650	4	9	10	.02814148
3	6	6	−.01411745	4	10	10	.05504858
3	6	7	.03516063				
3	6	8	.07516477	5	5	5	−.04386690
3	6	9	.04998029	5	5	6	.08650875
3	6	10	.00808104	5	5	7	.03500443
				5	5	8	−.01294078
3	7	7	−.00985944	5	5	9	.02870872
3	7	8	.02724675	5	5	10	.06403893
3	7	9	.06668946				
3	7	10	.05474855	5	6	6	−.03350414
				5	6	7	.06117194
3	8	8	−.00741466	5	6	8	.05170139
3	8	9	.02201080	5	6	9	−.00581844
3	8	10	.05915185	5	6	10	.00668642
3	9	9	−.00586119	5	7	7	−.02189833
3	9	10	.01830449	5	7	8	.04397838
				5	7	9	.05640500
3	10	10	−.00479475	5	7	10	.00564310
4	4	4	.12925988	5	8	8	−.01467602
4	4	5	−.02134197	5	8	9	.03341818
4	4	6	.02196251	5	8	10	.05551730
4	4	7	.08246301				
4	4	8	.05322789	5	9	9	−.01055417
4	4	9	.00474524	5	9	10	.02668086
4	4	10	−.00303408				
				5	10	10	−.00808528
4	5	5	.10673496				
4	5	6	.00196740	6	6	6	.09687151
4	5	7	−.00100286	6	6	7	−.02410491
4	5	8	.06117005	6	6	8	.02789589
4	5	9	.06583263	6	6	9	.06239762
4	5	10	.02450630	6	6	10	.01401545
4	6	6	.08660706				
4	6	7	.01580477	6	7	7	.08320655
4	6	8	−.00811483	6	7	8	−.00757894
4	6	9	.04086770	6	7	9	.00739178
4	6	10	.06525394	6	7	10	.05666489
4	7	7	.07367692	6	8	8	.06900813
4	7	8	.02281235	6	8	9	.00401100
4	7	9	−.00867086	6	8	10	−.00218959

TABLE I—*Continued*

r	s	t	C_{rst}	r	s	t	C_{rst}
6	9	9	.05911346	8	8	8	.07864342
6	9	10	.01077477	8	8	9	−.02407255
				8	8	10	.02947714
6	10	10	.05253399				
				8	9	9	.06917670
7	7	7	−.03776986	8	9	10	−.01154288
7	7	8	.07123199				
7	7	9	.01652253	8	10	10	.05829179
7	7	10	−.00951384				
				9	9	9	−.03353927
7	8	8	−.03035843	9	9	10	.06114106
7	8	9	.05351461				
7	8	10	.03227809				
				9	10	10	−.02786041
7	9	9	−.02107272				
7	9	10	.04012854				
				10	10	10	.06681991
7	10	10	−.01469083				

TABLE II

r	s	C_{rst}
0	0	$2\cdot 3^{-t-1}$
0	1	$(8t + 2)3^{-t-2}$
0	2	$(16t^2 + 2)3^{-t-3}$
0	3	$(64t^3 - 48t^2 + 56t + 6)3^{-t-5}$
1	1	$(32t^2 - 48t + 10)3^{-t-3}$
1	2	$(64t^3 - 240t^2 + 200t + 18)3^{-t-4}$
1	3	$(256t^4 - 1664t^3 + 3056t^2 - 1264t + 78)3^{-t-6}$
2	2	$(128t^4 - 1024t^3 + 2464t^2 - 1664t + 66)3^{-t-5}$
2	3	$(512t^5 - 6528t^4 + 28160t^3 - 46848t^2 + 24824t + 438)3^{-t-7}$
3	3	$(2048t^6 - 39936t^5 + 285824t^4 - 920832t^3 + 1314560t^2 - 647280t + 4410)3^{-t-9}$

An example of such an application will be given to illustrate the various technical problems involved.

The Blasius Equation.

$$y''' + yy'' = 0 \tag{52}$$

$$y(0) = y'(0) = 0; \qquad y'(\infty) = 2. \tag{53}$$

This is a well-known equation whose numerical solution has long been known [4]. However, as an example of the Laguerre functions method, we sought an approximate solution of the form

$$y = f_N(x) = 2x + a_N + \sum_{r=0}^{N} b_{Nr}\lambda_r \tag{54}$$

where a_N, b_{N0}, b_{N1}, $\cdots$, b_{NN} were constants to be determined.

The boundary condition at infinity is clearly satisfied, and, to attend to the conditions at $x = 0$, we need

$$a_N + \sum_{r=0}^{N} b_{Nr} = 0 \tag{55}$$

and

$$2 - \sum_{r=0}^{N} (r + \tfrac{1}{2}) b_{Nr} = 0 \tag{56}$$

thus leaving us the possibility of imposing N further conditions on the $(N + 2)$ coefficients a_N, b_{Nr} $(r = 0, 1, \cdots, N)$. As an obvious set of conditions, we should substitute from (54) in (52), express $y''' + yy''$ as a linear sum of λ_n's, and equate the coefficients of λ_0, λ_1, $\cdots$, λ_{N-1} to zero. However, there is a difficulty in the way of this program. The expression for λ_n' in terms of the Laguerre functions themselves involves all the λ_r for $0 \leqq r \leqq n$, while the expressions for λ_n'', λ_n''' become very complicated. The situation can be saved by pre-multiplying (52) by x^3 and making repeated use of the relation

$$x\lambda_n' = \tfrac{1}{2}(n + 1)\lambda_{n+1} - \tfrac{1}{2}\lambda_n - \tfrac{1}{2}n\lambda_{n-1}. \tag{57}$$

Proceeding in this way, one finds oneself confronted with expressions of the form $x^r\lambda_n$. They, too, can be resolved by the repeated use of the relation

$$x\lambda_n = -(n + 1)\lambda_{n+1} + (2n + 1)\lambda_n - n\lambda_{n-1}, \tag{58}$$

which also follows trivially from the fundamental properties of Laguerre polynomials.

After making all of these substitutions into (52), we still have to deal with products of Laguerre functions arising from the nonlinear term, and these have to be resolved by (3). We are now in a position to equate the coefficients of λ_0, λ_1, $\cdots$, λ_{N-1} to zero, obtaining a set of N quadratic equations which, together with (55) and (56), should suffice to determine the coefficients. In general, there will be more than one solution, and any one of them might, if N is large enough, be expected to yield a function $f_N(x)$ which will approximate the exact solution of (52). Since the solutions are obtained by equating the coefficients of λ_0, λ_1, $\cdots$, λ_{N-1} to zero, it has been found useful in practice to select the one for which the coefficient of λ_N is least in absolute value.

Setting up the equations even for so simple a case as (52) is not a trivial task, and can become extremely laborious for more complex equations. However, there would be no real difficulty in having all the work, including the formal steps represented by (3), (57), and (58), programmed for an electronic computing machine.

We have described the procedure so far in some detail, since it is of quite general application. The next step is actually to solve the equations for a_N, b_{Nr} $(r = 0, 1, \cdots, N)$, and for this purpose the following type of approach has been found to be practical. One first solves for some small value of N. The advance from N to $N + 1$ is effected by solving the equations for $N + 1$ by the Newton-Raphson method, taking as a first approximation to this solution

$$a_{N+1}^{(1)} = a_N, \quad b_{N+1,r}^{(1)} = b_{N,r} \quad (r = 0, 1, \cdots, N), \quad b_{N+1,N+1}^{(1)} = 0. \tag{59}$$

An advantage of this choice of first approximation is that it might, for obvious reasons, be expected to be better as N increases. Hence the number of steps required for convergence decreases.

In the particular case of the Blasius equation, we started with $N = 1$, i.e., with three coefficients to be determined. Eliminating two of them by (55), (56) left us with a quadratic equation in one unknown. The step to $N = 2$ was effected as described. There would be no difficulty in principle in carrying on to higher values of N. However, the arithmetic soon becomes extremely laborious, and the task is best handed over to an electronic computing machine. The result for $N = 2$ is shown in Table III. The function tabulated as $f(x)$ was obtained by direct numerical integration, using essentially Hamel's original method, and is given correct to three decimal places.

TABLE III

x	$f_2(x)$	$f(x)$
0	0	0
1	0.732	0.650
2	2.337	2.305
3	4.247	4.280
4	6.252	6.279
5	8.248	8.279
6	10.273	10.279

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The Behavior of Pseudo-Random Sequences Generated on Computers by the Multiplicative Congruential Method

By V. D. Barnett

1. Introduction. The use of pseudo-random elements in Monte Carlo work is essential when the scale of this work is such that the calculations involved are too extensive for hand calculating machines and it is necessary to employ an electronic computer. Although the ability of modern digital computers to perform simple binary operations at very high speed makes their use in this work particularly relevant, the limited extent of the computer memory, the relatively slow input speeds and speeds of access to the memory, and the very large number of random elements required (often of the order of 10^6) combine to make it unfeasible to prepare the elements beforehand in the required form for input to the computer (e.g., on tape or cards), and we must resort to some means of generation of the random elements within the computer.

Mechanical means of generation on peripheral equipment, e.g., by radioactive decay, thermal noise in electronic valves, etc., are undesirable because of the irreproducibility of the numbers obtained, which enables no check to be kept on their quality. It is therefore natural to employ some deterministic method of generation of the random elements by recurrence relationships. One such technique which has attracted much attention is the multiplicative congruential method (see for example [1], [2] and [3]) which proceeds as follows:

Choose ρ, x at random as the starting values.

Successive random elements are then obtained by the recurrence relationship

$$x_{r+1} \equiv x \cdot x_r \pmod{M} \tag{1}$$

with $x_0 = \rho$: that is, x_{r+1} is of the form $\rho x^r \pmod{M}$.

Using this method to generate pseudo-random sequences, one must place certain restrictions on x and ρ to ensure that the process does not degenerate to zero and that the maximum possible cycle of distinct elements is obtained. We shall see that only two restrictions must be placed on x and ρ to produce the maximum cycle and that it is also possible to describe fully the behavior of the system for x and ρ not satisfying these restrictions.

Because of the binary base of most digital computers, it is convenient to choose M of the form 2^k. Reduction, modulo M, is then simply a shift of the product ρx^r to retain the least significant k binary digits. If the computer, in common with many modern computers, has a facility for low multiplication, i.e., multiplication which retains only the least significant half of the set of binary digits comprising the product, and the numbers are stored as k_0 digits, it is a further advantage to choose

$$M = 2^{k_0},$$

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and this enables the successive random elements to be obtained by just one operation.

The length of the cycle of iterates and the conditions under which the maximum cycle is obtained must obviously depend on ρ, x, and $M = 2^k$, and although this system (for $M = 2^k$) is discussed in the literature [1], [4], [5], there appears to be some confusion as to the behavior of the system for given ρ, x, and M.

It has often been assumed desirable to choose x of the form

$$x = 5^{2k+1}$$

$$x = 7^{4k+1}$$

$$x = 13^{13} \text{ etc.}$$

to ensure the maximum cycle; see [3]. For instance, Taussky and Todd [1] state that for the system

$$\rho \equiv 1 \pmod 5$$

$$x = 5^{17}$$

$$M = 2^{42}$$

they obtain a cycle of length 2^{40}. We see immediately in this case that the cycle cannot be of length 2^{40}. As they demonstrate in the same paper, for an x in this form (i.e., $\equiv 5 \pmod 8$) the periods of successive digits from the least significant digit are as follows: 1, 1, 2, 4, 8, 16, $\cdots$, the least significant digit being always 1, the next always 0, the next alternately 0 or 1 the next 0, 0, 1, 1 and so on.

Now, if we choose ρ to be of the form 1 (mod 5), say $\rho = 256 = 2^8$, then the effect of multiplying the successive powers of x by 2^8 and reducing the product modulo 2^{42} is to shift each digit to a position 8 places more significant and to fill in the 8 least significant places with zeros. Because of the strict increase by factors of 2 in the period of the digits as they become more significant, this must result in a reduction of the length of the maximum cycle by the factor 2^8. (No formal proof of this maximum appears to have been given previously.)

This example should suffice to illustrate the importance of obtaining a set of necessary and sufficient conditions on ρ and x for the maximum cycle-length to be achieved, and also of giving a description of the behavior of the system when these conditions are not satisfied.

The only attempts to define formally the restrictions necessary on ρ and x are those of Leslie and Gower [4] and Certaine [5].

Leslie and Gower state that a maximum period of 2^{k-2} distinct random elements is obtained subject to:

(1) Choosing ρ and x' at random;

(2) Replacing x' by x the closest number to x' such that $x \equiv 5 \pmod 8$;

(3) Forming successively the numbers $\rho x^r \pmod{2^k}$, $r = 1, 2, \cdots$.

The random elements are then all 2^{k-2} numbers (mod 2^k) whose least significant binary digits are 10. This result is attributed to a theorem by Euler.

On examination we find that condition (2) on x is more restrictive than is necessary; and, as we have already seen, it is essential that some restriction be placed on ρ.

A discussion of the form of x necessary to ensure a cycle of maximum length and of the length of this maximum cycle has been given by Certaine [5] for general M, but his conclusions on the form of x for the particular case

$$M = 2^k$$

would again appear to be unnecessarily restrictive.

Furthermore, no attempt has been made to describe in what way the system is affected by a choice of x and ρ which do not satisfy the conditions required for the maximum cycle.

The system of numbers obtained from equation (1) for $M = 2^k$ is fully described in the following section, the proofs of the results being given in Section 3.

2. Formal Description of System of Numbers Generated by $x_{r+1} = x \cdot x_r \pmod{2^k}$; $\rho = x_0$. Under favorable conditions on x and ρ we obtain a maximum cycle of 2^{k-2} elements, all of which are distinct. The conditions on x and ρ to achieve this maximum cycle are:

I: $x \equiv \pm 5 \pmod 8$, i.e.

$x \equiv 5 \pmod 8$ or $x \equiv 3 \pmod 8$

II: $\rho \equiv 1 \pmod 2$, i.e.

ρ must be odd.

I and II are necessary and sufficient for the maximum cycle to be obtained.

Relaxation of these conditions affects the length of the cycle, and in some cases causes the process to degenerate to zero.

2.1. Relaxation of Condition II. If ρ is even and condition I is satisfied, the maximum cycle of 2^{k-2} distinct iterates is reduced in length by a factor 2^j where this is the highest power of two by which ρ is divisible, i.e., if $\rho \equiv 2^j \pmod{2^{j+1}}$ the cycle is of length 2^{k-j-2}.

We have already seen an illustration of this effect in the discussion of the system described by Taussky and Todd [1].

2.2. Relaxation of Condition I.

(a) If x is even, the maximum number of distinct iterates is k, generated by $x \equiv 2 \pmod 4$. In general, if $x \equiv 2^j \pmod{2^{j+1}}$ the number of distinct iterates is $\left[\frac{k}{j}\right]$ (where $[z]$ signifies the least integer greater than, or equal to, z). In all cases for x even the process degenerates to zero on the $\left[\frac{k}{j}\right]$th element produced. If ρ is even, say $\rho \equiv 2^l \bmod 2^{l+1}$, the process degenerates to zero on the $\left[\frac{k-l}{j}\right]$th element produced.

(b) If x is odd, the maximum cycle is generated, consisting of 2^{k-2} distinct iterates, if and only if

$$x \equiv 3 \text{ or } 5 \pmod 8.$$

For any other odd values of x the length of the cycle is decreased as follows:

If $x \equiv 7$ or $9 \pmod{16}$, length of cycle is 2^{k-3};

If $x \equiv 15$ or $17 \pmod{32}$, length of cycle is 2^{k-4}, etc.

We may completely specify all odd integers in this way, with the general result that if $x \equiv 2^j \pm 1 \pmod{2^{j+1}}$ the length of the cycle obtained is $2^{k-j}; j \geqq 2$; all iterates being distinct.

We may summarize these results as follows.

If $k = 2, 3$ the maximum number of distinct elements is k, generated by $x \equiv 2 \pmod 4$, $\rho \equiv 1 \pmod 2$, and the process degenerates to zero.

If $k > 3$ the maximum cycle is of 2^{k-2} distinct elements and is generated by $x \equiv 3$ or $5 \pmod 8$, again for $\rho \equiv 1 \pmod 2$.

Small values of k are, of course, of little practical interest in the generation of pseudo-random elements.

Relaxation of the conditions I and II simultaneously has the effect of combining the results of paragraphs (a) and (b). For example,

$$\left.\begin{aligned} \rho &\equiv 2 \pmod 4 \\ x &\equiv 7 \pmod{16} \end{aligned}\right\}$$

will produce a cycle of 2^{k-4} distinct elements, or, again,

$$\left.\begin{aligned} \rho &\equiv 2 \pmod 4 \\ x &\equiv 2 \pmod 4 \end{aligned}\right\}$$

will produce $\left(\left[\frac{k-1}{2}\right]\right)$ distinct elements, the process degenerating to zero.

3. Proof of Results of Section 2. Let us first consider condition I. For general x we have two possibilities:

x even: i.e., $x = 2m$

x odd: $x = 2m + 1$

3.1. x even. $M = 2^k$

Let $N(x) = [k/I(x)]$ where $I(x)$ is the index of the greatest power of 2 which divides x. Then

$$I(x^{N(x)}) \geqq k \text{ (since } I(x^m) = mI(x) \neq 0).$$

Hence $x^{N(x)} \equiv 0 \pmod{2^k}$ and number of distinct iterates cannot be greater than $N(x)$, for at this stage the process degenerates to zero.

Further, by the definition of $N(x)$, the process cannot terminate earlier.

Also, if $0 \leqq m, n \leqq N(x) \Rightarrow a^m \equiv a^n \pmod{2^k}$, then

$$a^{m-n} \equiv 1 \pmod{2^k}$$

and hence

$$m - n = 0.$$

Therefore, the congruence classes of the $N(x) + 1$ powers of x: $1, x, x^2, \cdots, x^{N(x)}$, are distinct and we must obtain $N(x)$ distinct iterates up to degeneration of the process (x^0 not being obtained). $N(x)$ will be a maximum for $N(x) = k$, i.e., $I(x) = 1$. So $x \equiv 2 \pmod 4$ generates k distinct elements, and in general $x \equiv 2^j \pmod{2^{j+1}}$ generates $\left[\frac{k}{j}\right]$.

3.2. x odd. $M = 2^k; x = 2m + 1.$

LEMMA. $Z - I(Z) \geqq 5$ *for* $Z \geqq 5$

Proof. If $2^i \leqq Z < 2^{i+1}$ then $I(Z) \leqq i$. Thus $Z - I(Z) \geqq 2^i - i$ which is monotone increasing in i and so $\geqq 5$ for $i \geqq 3$, i.e., for $Z \geqq 8$. Also if $Z = 5, 6, 7$ then $I(Z) = 0, 1, 0$ respectively. Hence $Z - I(Z) \geqq 5$, all $Z \geqq 5$ as required.

Now if $x = 2m + 1$, the 2^{k-1} congruence classes of x in the range $(0, 2^k)$ form a multiplicative group, the order $n(x)$ of the congruence class of x being the least integer such that

$$x^{n(x)} \equiv 1 \pmod{2^k},$$

hence, $n(x)$ divides 2^{k-1} [6].

Furthermore, the congruence classes of $x, x^2, \cdots, x^{n(x)}$ include all powers of x and are distinct, for if

$$x^m \equiv x^n \pmod{2^k} \qquad m > n, \quad \text{then}$$

$$x^{m-n} \equiv 1 \pmod{2^k},$$

and $m - n$ must be greater than $n(x)$.

So we see that the cycles generated by odd integers have length of a power of 2, the maximum cycle being given by the odd integer of greatest order in the group, and we must find the conditions necessary on x for it to have this greatest order. That the process must cycle is obvious, for if

$$x^j \equiv 0 \pmod{2^k}, \quad \text{then}$$

$$x \equiv 0 \pmod{2^k} \quad \text{since} \quad x \not\equiv 2^k$$

i. e., x is even, which is not true.

Now, $n(x)$ is determined by

$$(1 + 2m)^{2^l} \not\equiv 1 \pmod{2^k}$$

but $(1 + 2m)^{2^{l+1}} \equiv 1 \pmod{2^k}$, where $l = I(n(x)) - 1$.

Consider $(1 + 2m)^{2^l} \pmod{2^{l+p}}$ for suitable p.

If $p = 3$, then

$$I\left\{(2m)^r \binom{2^l}{r}\right\} \geqq r - I(r) + l$$

$$\geqq l + 5 \quad \text{for} \quad r \geqq 5.$$

Thus

$$(1 + 2m)^{2^l} \equiv \left\{\sum_0^4 (2m)^r \binom{2^l}{r}\right\} \pmod{2^{l+3}}, \tag{2}$$

and setting $\phi = (1 + 2m)^{2^l} - 1$, we have

$$\phi \equiv \left[2^{l+1}m + 2^{l+1}(2^l - 1)m^2 + \frac{2^{l+3}}{3}(2^l - 1)(2^{l-1} - 1)m^3 + \frac{2^{l+2}}{3}(2^l - 1)(2^{l-1} - 1)(2^l - 3)m^4\right] \pmod{2^{l+3}}.$$

Hence, if $l \geqq 2$

$$\begin{aligned}\phi &\equiv 2^{l+1}[m - m^2 - 2m^4] \pmod{2^{l+3}} \\ &\equiv 2^{l+1}[m(m+1) - 2m^2(m^2+1)] \pmod{2^{l+3}} \\ &\equiv 2^{l+1}m(m+1) \pmod{2^{l+3}}\end{aligned}$$

(for if m odd or even $2m^2(m^2 + 1) \equiv 0 \pmod 4$).

We must now distinguish between

$$\text{(i)}\ m \equiv 0, -1 \pmod 4$$

$$\text{(ii)}\ m \equiv 1, 2 \pmod 4.$$

(i) $(1 + 2m)^{2^l} - 1 \equiv 0 \pmod{2^{l+3}}$ hence, if $l + 3 = k$, $x^{M/8} \equiv 1 \pmod M$) so that $n(x)$ divides $M/8$.

(ii) $(1 + 2m)^{2^l} - 1 \equiv 0 \pmod{2^{l+2}}$ (for $m(m + 1) \equiv 2 \pmod 4$ but $(1 + 2m)^{2^l} - 1 \not\equiv 0 \pmod{2^{l+3}}$. Hence, taking $k = l + 2$; $k = l + 3$, we have $x^{M/4} \equiv 1$, $x^{M/8} \not\equiv 1$, (mod M), so that the maximum value of $n(x)$ is $M/4$, and it assumes this value for all $x = 1 + 2m$, where $m \equiv 1, 2 \pmod 4$, i.e.,

$$x \equiv 3 \pmod 8, \quad \text{or}$$

$$x \equiv 5 \pmod 8$$

and only these values.

Taking $p = 4$, we again find that

$$\phi \equiv m(m + 1)2^{l+1} \pmod{2^{l+4}} \qquad (l \geqq 3)$$

and we can immediately determine for what values of x we obtain the next largest cycle.

For this we want

$$m(m + 1) \equiv 0 \pmod 4, \quad \text{and}$$

$$m(m + 1) \not\equiv 0 \pmod 8, \quad \text{i.e.}$$

$$m \equiv 3, 4 \pmod 8, \quad \text{giving}$$

$$x \equiv 7, 9 \pmod{16}.$$

Similarly, we may extend this result by suitable choice of p to show that in general we obtain a maximum cycle of 2^{k-j} distinct elements for

$$x \equiv 2^j \pm 1 \pmod{2^{j+1}}.$$

This completely specifies the system for all odd x, for any odd value of x can be expressed in the form

$$x \equiv 2^j \pm 1 \pmod{2^{j+1}}, \qquad j = 2, 3, \cdots$$

($x = 1$ is, of course, trivial).

We now consider the effect of different values of ρ on these results. Any value of x which is of the form $x \equiv 2^j \pm 1 \pmod{2^{j+1}}$; $j \geq 2$, i.e., any odd value of x, produces a cycle of 2^{k-j} distinct elements having the following characteristics:

(a) The least significant digit is 1;

(b) The least significant $j + 2$ digits are of total order 4;

(c) The order of the $(j + 3)$th digit is 2, the order of the $(j + 4)$th digit is 4, etc.; that is, the order of the successive digits beyond the $(j + 2)$th have orders which increase by the factor 2 as they become increasingly more significant.

Therefore, it is apparent by inspection that the effect of multiplication of the elements by any odd value of ρ will be to leave these characteristics invariant and to unalter the length of the cycle of elements obtained.

Furthermore, if ρ is even, say $\rho \equiv 2^r \pmod{2^{r+1}}$, the r least significant digits must become zero and the above characteristics will then be true for the remaining $k - r$ digits, i.e., multiplication by an even-valued ρ has the effect of shifting the digits to a position r places more significant, performing a permutation of the digits remaining after such a shift, which does not affect their order, and substituting zeros for the least significant r digits. This must result in a reduction in the length of the cycle of elements by the factor 2^r.

These remarks are easily verified if we consider the effect of multiplication by ρ of the equation (2) for suitable choice of p for the form of x to be studied.

If, however, x is even, say $x \equiv 2^j \pmod{2^{j+1}}$, the successive powers of x have j, $2j$, $\cdots$ zeros as their least significant j, $2j$, $\cdots$ digits, and multiplication by any odd-valued ρ cannot affect the number of distinct iterates before degeneration, for it must ensure that in the successive iterates the $(j + 1)$th, $(2j + 1)$th$\cdots$ digits are non-zero.

When ρ is even, say $\rho \equiv 2^r \pmod{2^{r+1}}$, multiplication by ρ will introduce r further zeros in place of the r least significant non-zero digits and will therefore reduce the number of iterates to $\left[\frac{k-r}{j}\right]$.

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A Note on the Convergence of Alternating Direction Methods

By Milton Lees

1. Introduction. Convergence of the single-parameter alternating direction methods of Douglas, Peaceman, and Rachford [1], [2], [3] has been proved for a wide class of elliptic difference equations; see, for example, Birkhoff and Varga [4]. The proof consists in showing that a certain matrix, similar to the defining matrix, has spectral radius less than one. Since the defining matrices for these methods are symmetric only when they are induced by a proper discretization of Laplace's equation in a rectangular region, an estimate for their spectral radii does not imply a corresponding norm estimate for their rate of convergence.

The purpose of this note is to present another proof of the convergence of the two basic alternating direction methods which, at the same time, provides a norm estimate for their rate of convergence. First, for simplicity, we shall consider Laplace's equation in an arbitrary, bounded lattice region. We shall prove that it is possible to select the acceleration parameter so that a suitable norm of the error is reduced after each iteration by the amount $1 - ch + 0(h^2)$, where h is the uniform node spacing for the lattice, and $c > 0$ depends only on the minimal eigenvalue for the Laplace operator. We shall indicate only briefly the extension of these results to elliptic equations with variable coefficients.

Our proof of the convergence of the alternating direction methods was motivated by two considerations; first, the close resemblance of the iteration equations to parabolic difference equations, and second, an integral estimate technique for proving that, under suitable conditions, the solutions of parabolic differential equations decay exponentially.

2. Formulation of the Problem. Let $\mathfrak{L}$ denote the (uniform) lattice of side h determined by the nodes $(\alpha h, \beta h)$, α and β being integers, positive, negative, or zero. Denote by $\bar{S}$ any finite subset of $\mathfrak{L}$. In the usual way [5], we decompose $\bar{S}$ into two disjoint subsets: S, the interior nodes of $\bar{S}$, and ∂S, the boundary nodes of $\bar{S}$.

If Q is any subset of $\mathfrak{L}$, we denote by $\mathfrak{C}(Q)$ the linear space of all real-valued functions on $\mathfrak{L}$ whose support is contained in Q. (In general, the support of a function f is the closure of the set $\{p \mid f(p) \neq 0\}$.) Note that $\mathfrak{C}(S)$ is finite-dimensional with dimension equal to the number of nodes in S. The advantage of thus trivially extending functions on Q to all of $\mathfrak{L}$ will become apparent later.

We denote by Δ the usual "five-point" Laplace difference operator, i.e.,

$$\Delta u = u_{x\bar{x}} + u_{y\bar{y}}, \quad u \in \mathfrak{C}(Q),$$

where the subscripts x and y denote forward difference quotients, and $\bar{x}$ and $\bar{y}$ denote backward difference quotients [6].

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We shall be concerned, first, with the Dirichlet problem for the elliptic difference equation

$$\Delta u = f \quad \text{in} \quad S, \tag{1.1}$$

where $f \in \mathcal{C}(S)$. More precisely, we seek a function $u \in \mathcal{C}(\bar{S})$ that satisfies (1.1) and assumes arbitrary prescribed values on ∂S.

It is easy to see that this problem is equivalent to the problem of solving a linear system of algebraic equations of order equal to the number of nodes in S. By means of the familiar maximum principle for the difference operator Δ, one can show that this system of linear equations always has a unique solution. In this paper we shall be concerned with the two basic, single-parameter alternating direction methods for the inversion of this system of equations.

3. The Alternating Direction Methods. Let $w(0) \in \mathcal{C}(\bar{S})$ be a function that agrees with u on ∂S, and otherwise arbitrary in S; $w(0)$ will serve as the initial approximation to u. Now, for $n \geqq 1$, we determine two sequences of functions $\{w(n)\}$ and $\{w^*(n)\}$ in $\mathcal{C}(\bar{S})$ such that

$$w(n) = w^*(n) = u \quad \text{on} \quad \partial S, \tag{2.1}$$

and

$$\begin{aligned} &\text{(a)} \quad \rho^{-1}[w^*(n) - w(n-1)] = w^*_{x\bar{x}}(n) + w_{y\bar{y}}(n-1) - f \\ &\text{(b)} \quad \rho^{-1}[w(n) - w^*(n)] = w_{y\bar{y}}(n) - w_{y\bar{y}}(n-1) \end{aligned} \tag{2.2}$$

in S. Here $\rho > 0$, the so-called acceleration parameter, is to be selected later. These equations define what is usually referred to as the Douglas-Rachford alternating direction method for solving $\Delta u = f$.

In [3] it is shown that (2.1) and (2.2) determine uniquely (alternately) the functions $w^*(n)$ and $w(n)$ in terms of $w(n-1)$ and the values of u on ∂S. This operation taking $w(n-1)$ into $w(n)$ involves only the inversion of approximately $2\sqrt{p}$ tridiagonal matrices, where p is the number of nodes in S.

Following Douglas and Rachford, we eliminate from (2.2) the auxiliary function $w^*(n)$. Adding (2.2a) to (2.2b), we obtain the equation

$$Dw(n) = w^*_{x\bar{x}}(n) + w_{y\bar{y}}(n) - f, \tag{2.3}$$

where we have introduced the notation

$$Dw(n) = \rho^{-1}[w(n) - w(n-1)].$$

Solving (2.2b) for w^*, we get

$$w^*(n) = w(n) - \rho^2 D w_{y\bar{y}}(n).$$

Hence,

$$w^*_{x\bar{x}}(n) = w_{x\bar{x}}(n) - \rho^2 \Lambda D w(n),$$

where Λ is defined as follows:

$$\Lambda u = u_{x\bar{x}y\bar{y}}.$$

Putting the value just obtained for $w_{xx}^*(n)$ into (2.3), we find that w is a solution of the fifth-order difference equation

$$Dw = \Delta w - \rho^2 \Lambda D w - f, \quad \text{in} \quad S, \quad \text{for} \quad n \geqq 1. \tag{2.4}$$

The other alternating direction method, the Peaceman-Rachford method, is very similar to this; it is characterized by the equations

$$\rho^{-1}[w^*(n) - w(n-1)] = w_{xx}^*(n) + w_{yy}(n-1) - f$$

and

$$\rho^{-1}[w(n) - w^*(n)] = w_{xx}^*(n) + w_{yy}(n) - f.$$

After eliminating from these equations the auxiliary function $w^*(n)$, we find that w is a solution of the fifth-order difference equation

$$Dw = 2\Delta w - \rho \Delta D w - \rho^2 \Lambda D w - 2f \quad \text{in} \quad S, \quad \text{for} \quad n \geqq 1. \tag{2.5}$$

Our problem, now, is to show that $w(n) \to u$, as $n \to \infty$.

4. Preparation. We define on $\mathcal{C}(S)$ an inner product $\langle u, v \rangle$ as follows:

$$\langle u, v \rangle = h^2 \sum_{P \in S} u(P) v(P).$$

The norm $\langle v, v \rangle^{1/2}$ induced by this inner product will be denoted by $\| v \|$. We require two additional norms for $\mathcal{C}(S)$:

$$\| v \|_1 = (\| v_x \|^2 + \| v_y \|^2)^{1/2}$$

and

$$\| v \|_2 = (\| v \|^2 + \rho^2 \| v_{xy} \|^2)^{1/2}.$$

A subscript S will be placed on these quantities whenever we want to indicate that the sum is to be extended only over S.

In [7] it is proved that the minimal eigenvalue λ of the Laplace difference operator Δ, relative to S, can be characterized as follows:

$$\lambda = \inf_{0 \neq v \in \mathcal{C}(S)} \frac{\| v \|_{1,S}^2}{\| v \|^2}.$$

Since $\| v \|_{1,S} \leqq \| v \|_1$, it follows immediately that

$$\lambda \| v \|^2 \leqq \| v \|_1^2, \tag{3.1}$$

for all $v \in \mathcal{C}(S)$.

On the strength of this inequality, we now prove the

LEMMA 1. *Any function $v \in \mathcal{C}(S)$ satisfies the inequality*

$$\| v \|_1^2 \geqq \lambda_0 \| v \|_2^2, \tag{3.2}$$

where

$$\lambda_0 = \frac{\lambda}{1 + 2\lambda\rho^2 h^{-2}}. \tag{3.3}$$

Proof. We have from (3.1) and the definition of $\| v \|_2$ that

$$\| v \|_1^2 - \lambda \| v \|_2^2 \geqq -\lambda \rho^2 \| v_{xy} \|^2.$$

To this inequality we apply the elementary inequality [8]

$$h^2 \| v_{xy} \|^2 \leq 2 \| v \|_1^2,$$

valid for all $v \in \mathcal{C}(S)$, and we find that

$$\| v \|_1^2 - \lambda \| v \|_2^2 \geq -2\lambda\rho^2 h^{-2} \| v \|_1^2,$$

which is equivalent to (3.2).

5. Convergence of the Alternating Direction Methods. Let $u \in \mathcal{C}(S)$ be the solution of the Dirichlet problem for $\Delta u = f$ in S, and let $w(n)$ be the corresponding solution of the Douglas-Rachford equation (2.4). Then, by linearity, the error function $v(n) = u - w(n)$ belongs to the linear space $\mathcal{C}(S)$, and satisfies, for $n \geq 1$, the difference equation

$$Pv \equiv Dv - \Delta v + \rho^2 \Delta Dv = 0 \tag{4.1}$$

in S. Let $\phi(n) = (1 + \rho\mu)^{-n}$, where $\mu > 0$. One verifies immediately that ϕ is a solution of the difference equation

$$D\phi = -\mu\phi. \tag{4.2}$$

THEOREM 1. *If $\mu \leq \lambda_0$, then the error function $v(n)$ for the Douglas-Rachford method satisfies the inequalities*

$$\| v(n) \|_i \leq \phi(n) \| v(0) \|_i \qquad (i = 1, 2)$$

Proof. We set

$$M = 2\rho \sum_{\xi=1}^{n} \phi^{-2}(\xi) \langle v(\xi), Pv(\xi) \rangle. \tag{4.3}$$

Since $v(\xi) \in \mathcal{C}(S)$, we see that $M = 0$.

Introduce the function $z(\xi) \in \mathcal{C}(S)$ by means of the formula

$$v(\xi) = \phi(\xi) z(\xi). \tag{4.4}$$

Then we have that

$$Dv = \phi Dz + zD\phi - \rho D\phi \cdot Dz$$

which, in view of (4.2) becomes

$$Dv = \phi[aDz - \mu z],$$

where $a = \phi^{-1}(1) > 0$.

From this and (4.1) it follows that

$$Pv = \phi[aDz - \mu z - \Delta z + a\rho^2 \Delta Dz - \rho^2 \mu \Delta z] = \phi[aDz + Bz], \tag{4.5}$$

where B is defined in the obvious way.

Because of (4.5), (4.3) becomes

$$M = \rho \sum_{\xi=1}^{n} 2 \langle z(\xi), aDz(\xi) + Bz(\xi) \rangle. \tag{4.6}$$

We now recall from [8] the following quadratic difference identities

$$2\langle z, Dz\rangle = D\| z \|^2 + \rho\| Dz \|^2, \tag{4.7}$$

$$2\langle z, \Lambda Dz\rangle = D\| z_{\xi\eta} \|^2 + \rho\| Dz_{\xi\eta} \|^2, \tag{4.8}$$

$$\langle z, \Delta z\rangle = -\| z \|_1^2, \tag{4.9}$$

and

$$\langle z, \Lambda z\rangle = \| z_{\xi\eta} \|^2. \tag{4.10}$$

Using these identities in (4.6), we obtain, after some slight rearrangement, the identity

$$M = \rho \sum_{\xi=1}^{n} \{aD\| z \|_2^2 + 2(\| z \|_1^2 - \mu \| z \|_2^2) + a^2\| Dz \|_2^2 + \mu a\| Dz \|_1^2\}. \tag{4.11}$$

Since $\| z \|_1^2 \geqq \lambda_0\| z \|_2^2$, by Lemma 1, and $\mu \leqq \lambda_0$, we have from (4.11) that

$$M \geqq \rho \sum_{\xi=1}^{n} aD\| z \|_2^2 = a\| z(n) \|_2^2 - a\| z(0) \|_2^2.$$

But, since $M = 0$, we obtain from this and (4.4) that

$$\| v(n) \|_2 = \phi(n)\| z(n) \|_2 \leqq \phi(n)\| z(0) \|_2 = \phi(n)\| v(0) \|_2,$$

which proves the second inequality of the theorem.

Next, we set

$$N = \rho \sum_{\xi=1}^{n} \phi^{-2}(\xi)\| Pv(\xi) \|_s^2. \tag{4.12}$$

Then, by (4.4) and (4.5), (4.12) becomes

$$N = \rho \sum_{\xi=1}^{n} \| aDz(\xi) + Bz(\xi) \|_s^2.$$

But

$$\| aDz + Bz \|_s^2 = a^2\| Dz \|_s^2 + \| Bz \|_s^2 + 2a\langle Dz, Bz\rangle_S \geqq a^2\| Dz \|^2 + 2a\langle Dz, Bz\rangle,$$

where we have dropped the subscript S, because $Dz \in \mathcal{C}(S)$. Hence,

$$N \geqq \rho \sum_{\xi=1}^{n} \{a^2\| Dz \|^2 + 2a\langle Dz, Bz\rangle\}. \tag{4.13}$$

Using the identities (4.7)–(4.10) again, we obtain from (4.13) the inequality

$$N \geqq \rho \sum_{\xi=1}^{n} \{a^2\| Dz \|^2 + aD\| z \|_1^2 - \mu aD\| z \|_2^2 + a\rho\| Dz \|_1^2\}.$$

Therefore, by adding a combination of the above inequalities,

$$N + \mu M \geqq \rho a \sum_{\xi=1}^{n} D\| z \|_1^2.$$

Again, $N + \mu M = 0$, so that

$$\| z(n) \|_1 \leqq \| z(0) \|_1,$$

which, in view of (4.4), implies that

$$\| v(n) \|_1 \leqq \phi(n)\| v(0) \|_1.$$

This completes the proof of the theorem.

Since $\rho\mu > 0$, Theorem 1 implies that the Douglas-Rachford method is convergent, but we also have the stronger result:

THEOREM 2. *If $\mu = \lambda_0$ and $\rho = h/\sqrt{2\lambda}$, then the error function $v(n)$ for the Douglas-Rachford alternating direction method satisfies the inequalities*

$$\| v(n) \|_i \leqq \left(1 + \frac{\sqrt{2\lambda}}{4} h\right)^{-n} \| v(0) \|_i \qquad (i = 1, 2). \tag{4.14}$$

Proof. The inequality (4.14) follows from Theorem 1, by minimizing $\phi(n)$ as a function of ρ.

In an analogous fashion, we can prove

THEOREM 3. *If $\mu = 2\lambda_0(1 - \lambda_0\rho)^{-1}$ and $\rho = h/\sqrt{2\lambda}$, then the error function $v(n)$ for the Peaceman-Rachford alternating direction method satisfies the inequality*

$$\| v(n) \|_1 \leqq \left(\frac{4 - \sqrt{2\lambda}h}{4 + \sqrt{2\lambda}h}\right)^{n} \| v(0) \|_1 .$$

Note that the Peaceman-Rachford method has a smaller convergence factor.

The preceding results can be extended to more general elliptic difference equations, e.g., to those that come from elliptic differential equations of the form

$$\frac{\partial}{\partial x}\left(a(x, y)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(b(x, y)\frac{\partial u}{\partial y}\right) = f(x, y)$$

in a bounded, open set Ω. The functions a and b must belong to $c^1(\bar{\Omega})$. The main point that we wish to emphasize is that, because the inner product $\langle u, v \rangle$ is defined as a sum over all of $\mathfrak{L}$, one must be able to extend a and b in a continuously differentiable way to a slightly larger set $\omega \supset \bar{\Omega}$. This can be carried out, for instance, by means of the Whitney extension theorem [9], in a form given by Hörmander [10].

One can also show that similar results can be proved for the Douglas-Rachford method as applied to elliptic equations in n independent variables, although this is not the case with the Peaceman-Rachford method [8].

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Calculation of π to 100,000 Decimals

By Daniel Shanks and John W. Wrench, Jr.

1. Introduction. The following comparison of the previous calculations of π performed on electronic computers shows the rapid increase in computational speeds which has taken place.

Author		Machine	Date	Precision	Time
Reitwiesner	[1]	ENIAC	1949	2037D	70 hours
Nicholson & Jeenel	[2]	NORC	1954	3089D	13 min.
Felton	[3]	Pegasus	1958	10000D	33 hours
Genuys	[4]	IBM 704	1958	10000D	100 min.
Unpublished	[5]	IBM 704	1959	16167D	4.3 hours

All these computations, except Felton's, used Machin's formula:

$$\pi = 16 \tan^{-1} \tfrac{1}{5} - 4 \tan^{-1} \tfrac{1}{239}. \tag{1}$$

Other things being equal, that is, assuming the use of the same machine and the same program, an increase in precision by a factor f requires f times as much memory, and f^2 times as much machine time. For example, a hypothetical computation of π to 100,000D using Genuys' program would require 167 hours on an IBM 704 system and more than 38,000 words of core memory. However, since the latter is not available, the program would require modification, and this would extend the running time. Further, since the probability of a machine error would be more than 100 times that during Genuys' computation, prudence would require still other program modifications, and, therefore, still more machine time.

2. A New Program. We discuss here a computation of π to more than 100,000D, which required 8 hours 43 minutes on an IBM 7090 system. This increase in speed by a factor of about 20 is largely due to the increased speed of the 7090 (it is about 7 times as fast as a 704), but substantial gains were also obtained by programming changes.

The formula we used, namely,

$$\pi = 24 \tan^{-1} \tfrac{1}{8} + 8 \tan^{-1} \tfrac{1}{57} + 4 \tan^{-1} \tfrac{1}{239}, \tag{2}$$

is due to Störmer [6], and has not been previously used in high-precision computation. The computation time breaks down as follows:

$8 \tan^{-1} \frac{1}{57}$: 3 hours 7 minutes

$4 \tan^{-1} \frac{1}{239}$: 2 hours 20 minutes

$24 \tan^{-1} \frac{1}{8}$: 2 hours 34 minutes

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Conversion (binary-to-decimal): 42 minutes.

To obtain these favorable times, two devices were used.

a) Instead of evaluating the Taylor series

$$A \tan^{-1} \frac{1}{m} = \sum_{k=0}^{\infty} \frac{(-1)^k Am}{(2k+1)m^{2(k+1)}} \tag{3}$$

term by term, one may compute two terms at a time by using

$$A \tan^{-1} \frac{1}{m} = \sum_{k=0}^{\infty} \frac{Am[(4k+3)m^2 - (4k+1)]}{(16k^2+16k+3)m^{4(k+1)}}. \tag{4}$$

This substitutes 2 (multi-precision) divisions, 1 multiplication, and 1 addition for 4 divisions, 1 subtraction, and 1 addition, and this eliminates 27% of the computation time. Further improvement in this direction, i.e., three terms at a time, is not possible here, since the divisors in (4), m^4 and $(16k^2 + 16k + 3)$, already tend to fill a 7090 word, whose maximum numerical size is $2^{35} - 1$.

b) Since the 7090 is a binary machine, the (multi-precision) division by m^4 and the multiplication by $[(4k + 3)m^2 - (4k + 1)]$ may be replaced by a simple shifting operation for the case $m = 8$ in $24 \tan^{-1} \frac{1}{8}$. Had this not been done the 2 hours 34 minutes listed above would be, instead, 6 hours 7 minutes with the double-term formula, (4), or 8 hours 21 minutes with (3).

The program using (4) requires three blocks of storage, one for the current $Am/m^{4(k+1)}$, one for the current term, and one for the partial sum. Since a 7090 word is equivalent to 10.536 decimal digits, the working storage for an N-place π is $3N/10.536$. Therefore, a core memory of 32,768 easily suffices for a computation to more than 100,000D.

3. The Check. Inasmuch as such a computation requires *billions* of (arithmetical) operations, it is clear that a check is necessary. This was obtained with Gauss's formula [7]:

$$\pi = 48 \tan^{-1} \tfrac{1}{18} + 32 \tan^{-1} \tfrac{1}{57} - 20 \tan^{-1} \tfrac{1}{239}. \tag{5}$$

During the computation, using (2), the numbers

$$A = 8 \tan^{-1} \tfrac{1}{57}$$

and

$$B = 8 \tan^{-1} \tfrac{1}{57} + 4 \tan^{-1} \tfrac{1}{239}$$

were written on tape. At the start of the check A and B were read into memory and $9A - 5B = 32 \tan^{-1} \frac{1}{57} - 20 \tan^{-1} \frac{1}{239}$ was computed (in 1 second). To this difference was added the new number $48 \tan^{-1} \frac{1}{18}$, which was computed by (4) in 4 hours 22 minutes. The check, therefore, takes less time than the original run (8 hours 1 minute), and is perfectly valid, since an error in any of the four arctangents will lead to a discrepancy between the results of (2) and (5).

4. The Result. The run and the check were both made on July 29, 1961, and such a discrepancy did in fact occur. The two values of π, in binary form, were compared by the machine in $\frac{1}{8}$ second, and were found to agree to only 234,848 bits. This is equivalent to 70,695 decimal places. Subsequently the error was isolated. It

was found to have occurred in the computation of $24 \tan^{-1} \frac{1}{8}$. The second value of π, computed using (5), was therefore correct throughout. When $24 \tan^{-1} \frac{1}{8}$ was recomputed, the two values of π agreed up to the last word. This comprised 333,075 bits, or 100,265 decimal places. The first 100,000 of these are given here.

The computation was of such a character that it was known *a priori* that the term involving $\tan^{-1} \frac{1}{239}$ contains most of the round-off and truncation error, and consequently that we have the inequalities:

$$\pi \text{ computed by (2)} < \pi < \pi \text{ computed by (5)}.$$

Thus the check and the computation give upper and lower bounds respectively, and, to the extent that they agree, π is determined absolutely.

Care has been taken in the output routines—in writing and printing—and, since the reproduction here is photographic, we believe that this value is entirely free from error.

5. A Million Decimals? Can π be computed to 1,000,000 decimals with the computers of today? From the remarks in the first section we see that the program which we have described would require times of the order of *months*. But since the memory of a 7090 is too small, by a factor of ten, a modified program, which writes and reads partial results, would take longer still. One would really want a computer 100 times as fast, 100 times as reliable, and with a memory 10 times as large. No such machine now exists.

There are, of course, many other formulas similar to (1), (2), and (5), and other programming devices are also possible, but it seems unlikely that any such modification can lead to more than a rather small improvement.

Are there *entirely different* procedures? This is, of course, possible. We cite the following: compute $1/\pi$ and then take its reciprocal. This sounds fantastic, but, in fact, it can be faster than the use of equation (2). One can compute $1/\pi$ by Ramanujan's formula [8]:

$$(6) \qquad \frac{1}{\pi} = \frac{1}{4}\left(\frac{1123}{882} - \frac{22583}{882^3}\,\frac{1}{2} \cdot \frac{1 \cdot 3}{4^2} + \frac{44043}{882^5}\,\frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1 \cdot 3 \cdot 5 \cdot 7}{4^2 \cdot 8^2} - \cdots\right).$$

The first factors here are given by $(-1)^k (1123 + 21460k)$. A binary value of $1/\pi$ equivalent to 100,000D, can be computed on a 7090 using equation (6) in 6 hours instead of the 8 hours required for the application of equation (2).* To reciprocate this value of $1/\pi$ would take about 1 hour. Thus, we can reduce the time required by (2) by an hour. But unfortunately we lose our overlapping check, and, in any case, this small gain is quite inadequate for the present question.

One could hope for a theoretical approach to this question of optimization—a theory of the "*depth*" of numbers—but no such theory now exists. One can guess that e is not as "deep" as π,† but try to prove it!

Such a theory would, of course, take years to develop. In the meantime—say, in 5 to 7 years—such a computer as we suggested above (100 times as fast, 100 times as reliable, and with 10 times the memory) will, no doubt, become a reality. At that time a computation of π to 1,000,000D will not be difficult.

* We have computed $1/\pi$ by (6) to over 5000D in less than a minute.

† We have computed e on a 7090 to 100,265D by the obvious program. This takes 2.5 hours instead of the 8-hour run for π by (2).

6. Acknowledgement. We wish to thank the IBM Corporation for its generosity in allowing us to use, free of charge, the 7090 in the IBM Data Processing Center, New York, for the computation described above.

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1. G. REITWIESNER, "An ENIAC determination of π and e to more than 2000 decimal places," *MTAC*, v. 4, 1950, p. 11–15.

2. S. C. NICHOLSON & J. JEENEL, "Some comments on a NORC computation of π," *MTAC*, v. 9, 1955, p. 162–164.

3. G. E. FELTON, "Electronic computers and mathematicians," *Abbreviated Proceedings of the Oxford Mathematical Conference for Schoolteachers and Industrialists at Trinity College, Oxford, April 8–18, 1957*, p. 12–17, footnote p. 12–53. This published result is correct to only 7480D, as was established by Felton in a second calculation, using formula (5), completed in 1958 but apparently unpublished. For a detailed account of calculations of π see J. W. WRENCH, JR., "The evolution of extended decimal approximations to π," *The Mathematics Teacher*, v. 53, 1960, p. 644–650.

4. F. GENUYS, "Dix milles decimales de π," *Chiffres*, v. 1, 1958, p. 17–22.

5. This unpublished value of π to 16167D was computed on an IBM 704 system at the Commissariat à l'Energie Atomique in Paris, by means of the program of Genuys.

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7. C. F. GAUSS, *Werke*, Göttingen, 1863; 2nd ed., 1876, v. 2, p. 499–502.

8. S. RAMANUJAN, "Modular equations and approximations to π," *Quart. J. Pure Appl. Math.*, v. 45, 1914, p. 350–372; *Collected papers of Srinivasa Ramanujan*, Cambridge, 1927, p. 23–39.

Note added in proof, December 1, 1961. J. M. GERARD of IBM United Kingdom Limited, who was then unaware of the computation described above, computed π to 20,000 D on the 7090 in the London Data Centre on July 31, 1961. His program used Machin's formula, (1), and required 39 minutes running time. His result agrees with ours to that number of decimals.

PI = 3.+

1415926535	8979323846	2643383279	5028841971	6939937510	5820974944	5923078164	0628620899	8628034825	3421170679
8214808651	3282306647	0938446095	5058223172	5359408128	4811174502	8410270193	8521105559	6446229489	5493038196
4428810975	6659334461	2847564823	3786783165	2712019091	4564856692	3460348610	4543266482	1339360726	0249141273
7245870066	0631558817	4881520920	9628292540	9171536436	7892590360	0113305305	4882046652	1384146951	9415116094
3305727036	5759591953	0921861173	8193261179	3105118548	0744623799	6274956735	1885752724	8912279381	8301194912
9833673362	4406566430	8602139494	6395224737	1907021798	6094370277	0539217176	2931767523	8467481846	7669405132
0005681271	4526356082	7785771342	7577896091	7363717872	1468440901	2249534301	4654958537	1050792279	6892589235
4201995611	2129021960	8640344181	5981362977	4771309960	5187072113	4999999837	2978049951	0597317328	1609631859
5024459455	3469083026	4252230825	3344685035	2619311881	7101000313	7838752886	5875332083	8142061717	7669147303
5982534904	2875546873	1159562863	8823537875	9375195778	1857780532	1712268066	1300192787	6611195909	2164201989
3809525720	1065485863	2788659361	5338182796	8230301952	0353018529	6899577362	2599413891	2497217752	8347913151
5574857242	4541506959	5082953311	6861727855	8890750983	8175463746	4939319255	0604009277	0167113900	9848824012
8583616035	6370766010	4710181942	9555961989	4676783744	9448255379	7747268471	0404753464	6208046684	2590694912
9331367702	8989152104	7521620569	6602405803	8150193511	2533824300	3558764024	7496473263	9141992726	0426992279
6782354781	6360093417	2164121992	4586315030	2861829745	5570674983	8505494588	5869269956	9092721079	7509302955
3211653449	8720275596	0236480665	4991198818	3479775356	6369807426	5425278625	5181841757	4672890977	7727938000
8164706001	6145249192	1732172147	7235014144	1973568548	1613611573	5255213347	5741849468	4385233239	0739414333
4547762416	8625189835	6948556209	9219222184	2725502542	5688767179	0494601653	4668049886	2723279178	6085784383
8279679766	8145410095	3883786360	9506800642	2512520511	7392984896	0841284886	2694560424	1965285022	2106611863
0674427862	2039194945	0471237137	8696095636	4371917287	4677646575	7396241389	0865832645	9958133904	7802759009
9465764078	9512694683	9835259570	9825822620	5224894077	2671947826	8482601476	9909026401	3639443745	5305068203
4962524517	4939965143	1429809190	6592509372	2169646151	5709858387	4105978859	5977297549	8930161753	9284681382
6868386894	2774155991	8559252459	5395943104	9972524680	8459872736	4469584865	3836736222	6260991246	0805124388
4390451244	1365497627	8079771569	1435997700	1296160894	4169486855	5848406353	4220722258	2848864815	8456028506
0168427394	5226746767	8895252138	5225499546	6672782398	6456596116	3548862305	7745649803	5593634568	1743241125
1507606947	9451096596	0940252288	7971089314	5669136867	2287489405	6010150330	8617928680	9208747609	1782493858
9009714909	6759852613	6554978189	3129784821	6829989487	2265880485	7564014270	4775551323	7964145152	3746234364
5428584447	9526586782	1051141354	7357395231	1342716610	2135969536	2314429524	8493718711	0145765403	5902799344
0374200731	0578539062	1983874478	0847848968	3321445713	8687519435	0643021845	3191048481	0053706146	8067491927
8191197939	9520614196	6342875444	0643745123	7181921799	9839101591	9561814675	1426912397	4894090718	6494231961
5679452080	9514655022	5231603881	9301420937	6213785595	6638937787	0830390697	9207734672	2182562599	6615014215
0306803844	7734549202	6054146659	2520149744	2850732518	6660021324	3408819071	0486331734	6496514539	0579626856
1005508106	6587969981	6357473638	4052571459	102897064l	4011097120	6280439039	7595156771	5770042033	7869936007
2305587631	7635942187	3125147120	5329281918	2618612586	7321579198	4148488291	6447060957	5270695722	0917567116
7229109816	9091528017	3506712748	5832228718	3520935396	5725121083	5791513698	8209144421	0067510334	6711031412
6711136990	8658516398	3150197016	5151168517	1437657618	3515565088	4909989859	9823873455	2833163550	7647918535
8932261854	8963213293	3089857064	2046752590	7091548141	6549859461	6371802709	8199430992	4488957571	2828905923
2332609729	9712084433	5732654893	8239119325	9746366730	5836041428	1388303203	8249037589	8524374417	0291327656
1809377344	4030707469	2112019130	2033038019	7621101100	4492932151	6084244485	9637669838	9522868478	3123552658
2131449576	8572624334	4189303968	6426243410	7732269780	2807318915	4411010446	8232527162	0105265227	2111660396
6655730925	4711055785	3763466820	6531098965	2691862056	4769312570	5863566201	8558100729	3606598764	8611791045
3348850346	1136576867	5324944166	8039626579	7877185560	8455296541	2665408530	6143444318	5867697514	5661406800
7002378776	5913440171	2749470420	5622305389	9456131407	1127000407	8547332699	3908145466	4645880797	2708266830
6343285878	5698305235	8089330657	5740679545	7163775254	2021149557	6158140025	0126228594	1302164715	5097925923
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8530614228	8137585043	0633217518	2979866223	7172159160	7716692547	4873898665	4949450114	6540628433	6639379003
9769265672	1463853067	3609657120	9180763832	7166416274	8888007869	2560290228	4721040317	2118608204	1900042296
6171196377	9213375751	1495950156	6049631862	9472654736	4252308177	0367515906	7350235072	8354056704	0386743513
6222247715	8915049530	9844489333	0963408780	7693259939	7805419341	4473774418	4263129860	8099888687	4132604721

FROM 50010 TO 100000

5685162396	5864573021	4315981931	9514735381	2974167729	4786724229	2465436680	0980676928	2382806899	6400482435

6222247715 8915049530 9844489333 0963408780 7693259939 7805419341 4473774418 4263129860 8099888687 4132604721

5695162396 5864573021 6315981931 9516735381 2974167729 4786724229 2465436680 0980676928 2382806899 6400482435
4037014163 1496589794 0924323789 6907069779 4223625082 2168895738 3798623001 5937764716 5122893578 6015881617
5578297352 3344604281 5126272037 3431465319 7777416031 9906655418 7639792933 4419521541 3418994854 4473456738
3162499341 9131814809 2777710386 3877343177 2075456545 3220777092 1201905166 0962804909 2636019759 8828161332
3166636528 6193266863 3606273567 6303544776 2803504507 7723554710 5859548702 7908143562 4014517180 6246436267
9456127531 8134078330 3362542327 8394497538 2437205835 3114771199 2606381334 6776879695 9703098339 1307710987
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8350493163 1284045212 1925651798 0694113528 0131470130 4781643788 5185290928 5452011658 3934196562 1349143415
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2901618766 7952406163 4252257719 5429162991 9306455377 9914037340 4328752628 8896399587 9475729174 6426357455
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2784768472 6860849003 3770242429 1651300500 5168323364 3503895170 2989392233 4517220138 1280696501 1784408745
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4611996653 8581538420 5685338621 8672523340 2830871123 2827892125 0771262946 3229563989 8989358211 6745627010
2183564622 0134967151 8819097303 8119800497 3407239610 3685406643 1939509790 1906996395 5245300545 0580685501
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4460477464 9159950549 7374256269 0104903778 1986835938 1465741268 0492564879 8556145372 3478673303 9046883834
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9598470356 2226293486 0034158722 9805349896 5022629174 8788202734 2092222453 3985626476 6914905562 8425039127

5771028402 7998066365 8254889264 8802545661 0172967026 6407655904 2909945681 5065265305 3718294127 0336931378
5178609040 7086671149 6558343434 7693385781 7113864558 7367812301 4587687126 6034891390 9562009939 3610310291
6161528813 8437909904 2317473363 9480457593 1493140529 7634757481 1935670911 0137751721 0080315590 2485309066
9203767192 2033229094 3346768514 2214477379 3937517034 4366199104 0337511173 5471918550 4644902636 5512816228
8244625759 1633303910 7225383742 1821408835 0865739177 1509682887 4782656995 9957449066 1758344137 5223970968
3408005355 9849175417 3818839994 4697486762 6551658276 5843858845 3142775687 9002909517 0283529716 3445621296
4043523117 6006651012 4120065975 5851276178 5838292041 9748442360 8007193045 7618932349 2292796501 9875187212
7267507981 2554709589 0455635792 1221033346 6974992356 3025494780 2490114195 2123828153 0911407907 3860251522
7429958180 7247162591 6685451333 1239480494 7079119153 2673430282 4418604142 6363954800 0448002670 4962482017
9289647669 7583183271 3142517029 6923488962 7668440323 2609275249 6035799646 9256504936 8183609003 2380929345

9588970695 3653494060 3402166544 3755890045 6328822505 4525564056 4482465151 8754711962 1844396582 5337543885
6909411303 1509526179 3780029741 2076651479 3942590298 9695946995 5657612186 5619673378 6236256125 2163208628
6922210327 4889218654 3648022967 8070576561 5144632046 9279068212 0738837781 4233562823 6089632080 6822246801
2248261177 1858963814 0918390367 3672220888 3215137556 0037279839 4004152970 0287830766 7094447456 0134556417
2543709069 7939612257 1429894671 5435784687 8861444581 2314593571 9849225284 7160504922 1242470141 2147805734
5510500801 9086996033 0276347870 8108175450 1193071412 2339086639 3833952942 5786905076 4310063835 1983438934
1596131854 3475464955 6978103829 3097164651 4384070070 7360411237 3599843452 2516105070 2705623526 6012764848
3084076118 3013052793 2054276428 6540360367 4532865105 7065874882 2569815793 6789766974 2205750596 8344086973
5020141020 6723585020 0724522563 2651341055 9240190274 2162484391 4035998953 5394590944 0704691209 1409387001
2645600162 3742880210 9276457931 0657922955 2498872758 4610126483 6999892256 9596881592 0560010165 5256375678

FROM 10001D TO 15000D

5667227966	1988578279	4848855834	3975187445	4551296563	4434803966	4205579829	3680435220	2770984294	2325330225
7634180703	9476994159	7915945300	6975214829	3366555661	5678736400	5366656416	5473217043	9035213295	4352916941
4599041608	7532018683	7937023488	8689479151	0716378529	0233529244	0773659495	6305100742	1087142613	4974595615
1384987137	5704710178	7957310422	9690666702	1449863746	4595280824	3694457897	7233004876	4765241339	0759204340
1963403911	4732023380	7150952220	1068256342	7471646024	3354400515	2126693249	3419673977	0415956837	5355516673
0273900749	7297363549	6453328886	9844061196	4961627734	4951827369	5583220757	3551766515	8985519098	6665393549
4810688732	0685990754	0792342402	3009259007	0173196036	2254756478	9406475483	4664776041	1463233905	6513433068
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FROM 15001D TO 20000D

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8172932393	0106766386	8240401113	0402470073	5085782872	4627134946	3685318154	6969046696	8693925472	5194139929
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FROM 20001D TO 25000D

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FROM 25001D TO 30000D

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FROM 25001D TO 30000D

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FROM 30001D TO 35000D

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FROM 35001D TO 40000D

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FROM 40001D TO 45000D

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4301650586	0296521745	9581988878	6804081103	2843273986	7198621306	2055598552	6603640504	6282152306	1545944744
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FROM 45001D TO 50000D

3874750758	0687653764	4578252443	2638046143	0428892359	3485296105	8269382103	4980004052	4840708440	3561167817
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4301650586	0296521745	9581988878	6804081103	2843273986	7198621306	2055598552	6603640504	6282152306	1545944744
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FROM 450010 TO 500000

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9328672547	4054365271	7595893500	7163360763	2161472581	5407642053	0200453401	8357233829	2661915308	3540951202
2632916505	4426123619	1970516138	3935732669	3760156914	4299449437	4485680977	5696303129	5887191611	2929468188
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1848630254	5016209298	0582920833	4813638405	4217200561	2198935366	9371336733	3924644161	2522319694	3471206417
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8020607820	5544629917	2320202822	2914886959	3997299742	9747115537	1858924238	4938558585	9540743810	4882624648
7880533042	7146301194	1589896328	7926783273	2245610385	2197011130	4665871005	0008328517	7311776489	7352309266
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FROM 50001D TO 55000D

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66203068880	52005559910	21233563710	82269225310	78145879250	93750441440	89339816080	65790087610	65024635190	70458288950
48179375660	81046474610	41051424980	87025213990	36870509370	23054477340	11264135480	92806481050	91077166770	82123833280
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16657176230	18247216830	50678076480	75734204910	55762821750	83972975130	44789909660	58953254890	40335615610	31674032760
47246921250	05759116250	15296545680	54466334980	14317670250	72956618440	77548746930	78462423730	72369819200	66204851180
94378866220	48072793520	02251079650	45334757270	41639107910	97295295080	12942922200	53477173040	18447791560	73991738410
83117103620	52439571610	52714669000	58147000020	63301045260	43547865900	32907332050	46833887200	78735444760	26479259970

FROM 55001D TO 60000D

69017091200	07874183730	67350877130	37697768340	96344252410	99499513880	31507487750	37433849450	82597655600	99655595430
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FROM 55001D TO 60000D

6901709120	0787418373	6735087713	3769776834	9634425241	9949951388	3150748775	3743384945	8259765560	9965559543
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5150789085	0712659970	3670872669	2764308377	2296859851	6912230503	7462744310	8529343052	7307886528	3977336246
0174635277	0320593817	9125396915	6210636376	2588293757	1373840754	4064689647	8310070458	0613446731	2715911946
0843593582	5987782835	2665311510	6504162329	5329047772	1740835593	4972375855	2138048305	0900096466	7608830154
0612824308	7406455944	3185341375	5220166305	8121110334	5312074508	6824339432	1590435944	3031243122	7471385842
0303901060	7094031523	5556172767	9941600203	9397509989	7629335325	8555756248	0899669182	9864222677	5023601932
5797472674	2578211119	7347094023	5745722227	1212526852	3842958742	7350156366	0093188045	4933389897	4157149054
4182559738	0808715652	8143010267	0460284316	8192303925	3529779576	5862414392	7015497408	7927313105	1636119137
5770089295	6482332364	8298263024	6079758757	6774537716	0102490804	6243018565	2416175665	5600160859	1215345562
6760219268	9982855377	8725831451	4408265458	3484409478	4631787773	7479465358	0169960779	4055687011	9232860804
1130904629	3508718271	2593466871	2766694873	8998245985	2778649956	9165464029	4589350649	6433580982	4765965165
1420909867	5520380830	9203230487	3427034682	8875160407	7546653834	6196112230	1375945157	9252696743	6425319273
9003603860	8236450762	6988274976	1872357547	6762889950	7521148048	5252795084	5033958570	8381304769	3788132112
3674281319	4879502280	6632017002	2460331989	6719706491	6374117585	4851878484	0120548446	7258885140	1562725019
8217190669	6081262778	5485964818	3696214107	2171421498	6361918774	7545096503	0895709947	0934337856	9816744658
2826791194	0611956037	8453978558	3924076127	6344105766	7510243075	5981455278	6167815949	6570625597	5507430652
1085301597	9080733437	3607943286	6757890533	4836695554	8680391343	3720156498	8342208933	9997164147	9746938696
9054800891	9306713805	7171505857	3071488156	4992071408	6758259602	8760564597	8242377024	2469805328	0566327870
4192676846	7116266879	4634869504	6450742021	9373945259	2626686135	5294062478	1361206202	6364981999	9949840514
3868285258	9563422643	2870766329	9304891723	4007254717	6418868535	1372332667	8779217383	4754148002	2803392997
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0646053323	8537203143	2421126074	2448584509	4580494081	8209276391	4000854042	2023556260	2185643489	9414543995
0410980591	8179488826	2805206644	1086319001	6885681551	6922948620	3010738897	1810077092	9059048074	9092427141
0189335428	1842999598	8169660993	8369616443	8152887721	4085268088	7574882932	5873580990	5670755817	0179491619
0611400190	8553744882	7262009366	8560447559	6557476485	6740081773	8170330738	0305476973	6097865438	5938218722
0583902344	4435088674	9986650604	0645874346	0053318274	3629617786	2518081893	1443632512	0510709469	0813586440
5192295129	3245007883	3398788429	3393424351	2634336520	4385812912	8343452973	0865290978	3300671261	7981303167
9438553572	6296998740	3595704584	5223085639	0098913179	4759487521	2639707837	5944861139	4519602867	5121056163
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4272031892	7706047177	8452793916	0472281534	3798035396	7986142437	0956683221	4914654380	1459382927	7393396032
7540480095	5223181666	7380357183	9327570771	4204672383	8624617803	9762923771	3120958078	9363841447	9298025880
6552212926	2093623930	6373134966	4018661951	0811583471	1733120258	0586672763	9992763579	0780638188	1306915636
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9527669435	6837676136	8711962921	8187545760	8161705303	1590728828	7007123136	6630872275	4918661395	7737305460
6599743781	0987649802	4140112421	4277366808	2751390959	3134041558	2626678951	0846776118	6659576601	6599817808
9414985754	9762843878	5610026379	6543178313	6340251358	1416115190	2096499133	5487331311	1502270068	1930135929
5959716401	9719605362	5033558479	9809634887	1803911161	2813595968	5654788683	2585643789	6173159762	0024196215
5289629790	4819822199	4622694871	3746244472	9093456470	0285376949	5885959160	6789282491	0544125159	9630078136
8367490209	3749157328	9627002865	6829344431	3423473512	3929825916	6739503425	9958689706	9726733258	2735903121
2887466604	5146148785	0346142827	7659916080	9039865257	5717263081	8334944418	2019353338	5071292345	7743755793
4406217871	1330063106	0033240539	9169368260	3746176638	5657588775	8020122936	6353270267	1006812618	2517291460
8202541892	8859352444	9107013820	6211553827	7935652969	1457650204	8643282865	5579347072	0963480737	2692141186
8954673227	6775133569	0190153723	6690368653	8916129168	8887876407	5254934942	4973342718	1178892759	9315967193
5475898809	7924525262	3636590363	2007085444	0784544797	3482918020	8204492667	0634420437	5553250505	2752283377
8887040804	0335319234	0768563010	9347772125	6390886404	1310107381	7853338316	0381352808	2811904083	2564401842
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FROM 60001D TO 65000D

0333152274	8177769528	4368163008	8531769694	7836905806	7106482808	3598046698	8410981351	5865490693	3319522394
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9280539940	9108754939	1660957316	1971503316	6968309929	4663491427	9878084225	7220697148	8755806374	8030886299
5118473187	1247772919	1007022758	8893486939	4562895158	0296537215	0409603107	7612898312	6358996489	3410247036
0366450586	8728758905	1406841238	1242473863	8542790828	2733827973	3268855049	3587430316	0274749063	1295723497
4261122151	7417153133	6186224109	1386950068	8835898962	3492763173	1647834007	7460886655	5987333821	1382992877
6911495492	1841920877	7160606847	2874673681	8861675072	2101726110	3830671787	8566948129	4878504894	3063086169
9487987031	6051588410	8282351274	1535385133	6589533294	8629494495	0618685147	7910580469	6039069372	6626703865
1290520113	7810858616	1888869479	5760741358	5534585151	7680519733	3443349523	0120395770	7396237713	1603024288
7200537320	9982530089	7761897312	9817881944	6717311606	4723147624	8457551928	7327828251	2718244680	7824215216

4695678192	9409823892	6284943760	2488522790	0362021938	6696482215	6280936053	7317804086	3727268426	6964219299
4681921490	8701707533	3610947913	8180406328	7387593848	2695355830	7739576144	7997270003	4728801827	8528138950
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4424164047	7503153605	4906591430	0781583724	3012313751	1562284015	8386442708	9071828481	6757527123	8467824595
3433444962	2010096071	0513706084	6180118754	3120725491	3349942476	1711563332	1408934609	1565615506	0031738421
8701570226	1031019166	0388706466	1438897736	3187809407	1152752817	4689576401	5810470169	6524755774	0891644568
6777171585	0058326994	3401677202	1567677240	6812836656	5264122982	4394651331	9735919970	9403275938	5026695574
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5732165554	9883421218	0284691252	9086101485	5278152776	2562375045	6375769497	7343368460	1560772703	5509629049
3924870884	0628106794	3622418704	7470083688	4267102255	8302403599	8416459511	2248527263	3632645114	0173952480

8619463584	0783753556	8856223171	1552094722	3065437092	6067973510	0056554938	1224575483	7285457117	9739361575
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4363308730	1398142114	3030600873	0666164803	6789840913	3592629340	2304324974	9268878316	4360268101	4309570716
1419128306	8657732353	2639653677	3903176613	6131596555	3584999398	6005651559	2193675997	7717933019	7446881483
7110320650	3693192894	5214026509	1546518430	9936553493	3371834252	9843367991	5939417466	2239003895	2767381333
0617747629	5749438687	1697845376	7219493506	5908757119	1772087547	7107189937	9608947745	1265475750	1871194870
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2264784216	0631227696	4696715647	3999043715	9033239065	6072664411	6438605404	8388471619	1210900870	1019130726
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1538094771	4601056333	4469988208	2212058728	1510729182	9712119178	7642488035	4672316916	5418522567	2923442918
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1998184590	2595835598	9260553299	6737704917	2245493532	9683300002	2301815172	2657578752	4058832249	0858212800
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7537792699	0695871212	8928801905	2031601285	8618254944	1335382078	4883465311	6326504076	4242839087	0121015194

2319616522	6842200371	1230464300	6734420647	4771802135	3070124098	8603533991	5266792387	1101706221	8658835737
8121093517	9775604425	6346949997	8725112544	0854522274	8109148743	0725986960	2040275941	1789425812	8188215995
2359658979	1811440776	5335432175	7595255536	1581280011	6384672031	9346507296	8079907939	6371496177	4312119402
0212975731	2516525376	8017359101	5573381537	7200195244	4543620071	8484756634	1540744232	8621060997	6132434875
4884743453	9665981338	7174660930	2053507027	1952983943	2714253711	5576660002	5784423031	0734295515	3394506048
6222764966	6876240793	2435319299	2639253731	0768921353	5257232108	0889819339	1686682789	4828117047	2624501948
4097009757	6092098372	4090074717	9733407881	4182519584	2598096241	7476101382	5264395513	5259311885	0456362641
8830033853	9652435997	4169313228	9471987830	8427600401	3680747039	0409723847	3945834896	1865397905	9411859931
0356168436	8692194853	8205578039	5773881360	6795499000	8512325944	2529724486	6667668346	4140218991	5944565309
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FROM 650010 TO 700000

9511911783	6587662528	0836900532	490045974 1	0947068772	9123282143	0463533728	3519953648	2743258331	1914445901
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3054262539	9583389794	2971602070	3387678150	3301028012	0095997252	2222808014	2357109476	0351925544	4349299867
6781789104	5559063015	9538097618	7592035893	7341978962	3589311259	8390259831	0267193304	1892151096	8915622506
9659119828	3234555030	5908173073	5195503721	6658702880	5399213857	6037035377	1051780212	8012956684	1984140362
8727256232	1442875430	2210094742	7210734741	3497551419	0737043318	2766261772	7599688882	6097225247	1336833534
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9912016659	6224360802	4287002454	7362036394	8412559548	8172727247	3653467783	6472019183	0399871762	7037515724
6499222894	6793232269	3619177641	6146187956	1395669956	7783068290	3165896994	3076733350	8234990790	6241002025
0613405734	4300695745	4746821756	9044165154	0636584680	4636926212	7421107539	9042188716	1276177870	1425886482
5775223889	1845995233	7629237791	5585744549	4773612955	2595222657	8636462118	3775984737	0034797140	8206994145
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1142066390	7459215273	3094068236	4944159089	1009220296	6805233252	6619891131	1842016291	6310768940	8472364436
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7056103973	9506264035	5442275156	1011072617	7937063472	3804990666	9221619711	9425912044	5084641746	3835899382
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1786671153	9195135769	6104864922	4900834467	1548638305	4477914330	0976804868	7834818467	2733758436	8927243104
4740680768	5278625585	1650920882	6381323362	3148733336	7147645204	5087662761	4950389949	5048095604	6098960432
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2141361982	3495159601	0647527125	7593518530	4332875537	7830575095	6742544268	4712219618	7091785607	8393614451
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6567277124	0903387727	7456290971	1013488518	4374118695	6554497457	3684521806	6982911045	0580042998	8795389902
7804383596	2824094218	6055628778	8428802127	5538848037	2864001944	1614257499	9042720095	9520465417	0598104989
9675045119	3647117277	2220436102	6140797508	0968697517	6600237187	7483480161	2031023468	0567112644	7661237476
2785219024	1202569943	5347162266	6089367521	9833111813	5111465038	5489502512	0655772636	1454736044	2685949807
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3668884462	8265133798	4916678394	0976135376	6251798258	2496634587	7195012438	4040359140	8492097337	5464247448
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2256345704	9731224526	9354193837	0048431833	5719651662	6721575524	1934019330	9901831930	9196582920	9696562476
6768365964	7019595754	7393455143	3741370876	1517323677	2042273856	7427917069	8204549953	0959188724	3493952409
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FROM 70001D TO 75000D

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FROM 75001D TO 80000D

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9762452056	1985576257	6000870817	3083288344	3818310700	5451449354	5885422678	5785519153	7229237955	5494333410
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FROM 80001D TO 85000D

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3617740078	3778342370	9152644063	0540718078	4335806107	2961105550	0204151316	9637304684	9213356837	2654003075
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FROM 85001D TO 90000D

7631857570	3810944339	0564564468	5241830281	4810799837	6918512127	2019350440	4180460472	1626939445	7883770901
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FROM 850010 TO 900000

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FROM 90001D TO 95000D

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FROM 95001D TO 100000D

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FROM 95001D TO 100000D

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975 455775 6843681110 1853174981 6701642664 7884090262 6828244482 5802753209 4549915104 5185177165 4631180490

4567985713 2575281179 1365627815 8111288816 5622858760 3087597496 3849435275 6766121689 5926148503 0785362045
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1834872587 4239194108 9044411595 9932760044 5065562064 6116465566 5487594247 3692523369 5599303035 5095817626
1762318495 6190649483 9673002037 7638743693 4399982943 0209147073 6189479326 9276244518 6560239559 0537051289
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TECHNICAL NOTES AND SHORT PAPERS

Characteristic Exponents of Mathieu Functions

By T. Tamir

The study of the properties and solutions of the Mathieu second-order differential equation has been very extensive in the last decades, due to the special interest presented by physical problems involving periodic media and problems separable in elliptic coordinate systems, [1], [2]. The purely periodic solutions of the Mathieu equation have been extensively tabulated with high accuracies over large ranges of the parameters involved [3], [4].

On the other hand, numerical tables for the non-periodic Floquet-type solutions are scarce [5], [6], [7]; in addition, the increments between two successive tabulated values are relatively large for some of these tables and the accuracy is relatively low in the other tables. Consequently, the applicability of the available data is rather limited.

The regions of "stable" solutions of the Mathieu equation are of particular interest in applications involving the wave equation with periodic boundary conditions, since these regions correspond to the propagating waves in the corresponding media. The first three regions of stable solutions were tabulated with this type of problem in mind, the numerical results being given in Tables I, II, and III.

The canonical form of the Mathieu equation considered is given by

$$y'' + (p - 2q \cos 2z)y = 0, \tag{1}$$

with p and q real. Every non-periodic solution of (1) is a linear combination of two linearly independent Floquet-type solutions of the form

$$y = e^{i\nu z}P(z), \tag{2}$$

where $P(z)$ is a periodic function of z, and ν is the so-called characteristic exponent. This exponent is a function of the parameters p and q only. When ν is real, the solutions (2) are called "stable" since they are uniformly bounded for all real z; then $\nu = f(p, q)$ may be chosen to be single-valued. (*Note*: In the tables and curves shown here, ν is *not* reduced to $0 \leqq \nu \leqq 1$ as is usually done, but its actual value is taken, thus defining the particular stable region it represents.)

It is shown in the references that all values of ν are solutions of the continued fraction equation

$$D_0 = \frac{1}{|D_1} - \frac{1}{|D_2} - \frac{1}{|D_3} - \cdots + \frac{1}{|D_{-1}} - \frac{1}{|D_{-2}} - \frac{1}{|D_{-3}} - \cdots \tag{3}$$

where

$$D_m = \frac{(2m + \nu)^2 - p}{q}, \qquad (m = 0, \pm 1, \pm 2 \cdots). \tag{4}$$

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For computation purposes, it is convenient to define

$$x = \nu^2 - p \tag{5}$$

so that equation (3) will take the form

$$x = \frac{q^2}{4(1+\nu)+x} - \frac{q^2}{|8(2+\nu)+x} - \cdots - \frac{q^2}{|4n(n+\nu)+x} - \cdots$$
$$+ \frac{q^2}{4(1-\nu)+x} - \frac{q^2}{|8(2-\nu)+x} - \cdots - \frac{q^2}{|4n(n-\nu)+x} - \cdots \tag{6}$$
$$(n = 1, 2, 3, \cdots)$$

The continued fractions are convergent since they satisfy the convergence test [2] which requires that $|D_n| \geqq 2$ for $n > N$, where N is a finite integer.

To find the roots of equation (6), it is noted that this equation is already expressed in a suitable form for solution by means of an iteration process, with x regarded as the unknown. Taking q and ν as variable parameters, such an iteration procedure was programmed on a computer which obtained p with an accuracy of 10^{-6}. As is proved in the Appendix, the iteration process converges in an alternating fashion, i.e., the exact result always lies between the values obtained by two successive iterations; hence, the iteration process must be carried out only until the difference between two successive results is less than the maximum desired deviation.

It was also noted that the number of iterations required to solve equation (6) was extremely large for a large range of parameters if a simple iteration procedure was used. This was due to a very slow convergence of the iteration process in some regions of q and ν. To improve this situation, the computer was programmed to differentiate between two cases:

a) "Fast" convergence: x'_{n+1} contained within the interval (x_n, x_{n+2}),

b) "Slow" convergence: x'_{n+1} outside the interval (x_n, x_{n+2}), where x_{n+1} is the result of the nth iteration, using x_n as the trial value; x'_{n+1} is the arithmetic mean of x_{n+1} and x_n.

After using the test to determine whether the convergence is "fast" or "slow," as defined above, the computer used either x_{n+2} or x'_{n+2}, respectively, as the next trial value for the $n+2$ iteration. This procedure reduced the number of iterations required from more than a thousand to less than twenty for the most slowly converging cases.

The results thus obtained are given in the appended tables, while Figure 1 shows these values plotted in the $p - q$ plane. Note that, for greater clarity, $P = \pm\sqrt{|p|}$ rather than p is plotted in the figure. Thus $p = P^2$ if $P \geqq 0$; $p = -P^2$ if $P < 0$.

Appendix. It is shown below that an iteration process based on equation (6), if converging, will do so in an alternating manner; i.e., the value of the exact solution always lies between the values of the results of any two successive iterations.

The iteration process may be written in the form

$$x_{n+1} = R_0^+(x_n) + R_0^-(x_n)$$

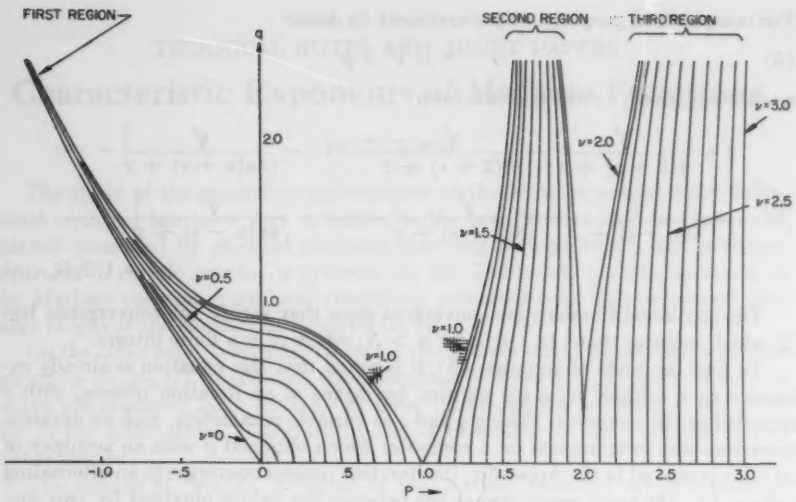

FIG. 1—Curves of q vs P for constant values of ν. The values of ν pertaining to each curve are given directly by the interceptions with the P axis. $p = P^2$ if $P \geqq 0$; $p = -P^2$ if $P < 0$.

where

$$R_0^{\pm}(x_n) = \frac{1}{a_0^{\pm}x_n + b_0^{\pm}} - \frac{1}{|a_1^{\pm}x_n + b_1^{\pm}} - \cdots - \frac{1}{|a_m^{\pm}x_n + b_m^{\pm}} - \cdots .$$

Then

$$R_m^{\pm}(x_n) = \frac{1}{a_m^{\pm}x_n + b_m^{\pm}} - \frac{1}{|a_{m+1}^{\pm}x_n + b_{m+1}^{\pm}} - \cdots$$

is a "remainder" function or the "tail" of a continued fraction after the first m terms have been omitted.

Disregarding the $\pm$ sign, we have

$$R_0(x_n) = \frac{1}{a_0 x_n + b_0 - R_1(x_n)}.$$

Differentiating with respect to x_n, we obtain

$$R_0'(x_n) = -\frac{a_0 - R_1'(x_n)}{[a_1 x_n + b_1 - R_1(x_n)]^2} = -R_0^2[a_0 - R_1'(x_n)].$$

Noting that $R_m(x_n)$ has the same functional form as $R_0(x_n)$, one finds

$$R_0'(x_n) = -[a_0R_0^2 + a_1(R_0R_1)^2 + a_2(R_0R_1R_2)^2 + \cdots].$$

In the case considered, $a_m = \frac{1}{q^2}$ or 1; hence $a_m > 0$. $R_m(x_n) \to 0$ as $m \to \infty$ so that $R_0'(x_n)$ has a finite negative value for any finite x_n. Consequently, the derivative of x_{n+1} is finite and negative so that the iteration process based on equation (6), if convergent, will converge in an alternating fashion.

TABLE I

Values of $p = f(q, \nu)$ for the first "stable" region (area between be_0 and bo_1 of Reference 3, p. xliv).

q	ν										
	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
0.1	−0.00500	0.00496	0.03480	0.08451	0.15406	0.24335	0.35222	0.48026	0.62631	0.78509	0.89877
0.2	−0.01991	−0.01011	0.01927	0.06815	0.13636	0.22359	0.32920	0.45175	0.58734	0.72206	0.79512
0.3	−0.04457	−0.03500	−0.00637	0.04117	0.10727	0.19126	0.29186	0.40633	0.52799	0.63820	0.68917
0.4	−0.07865	−0.06940	−0.04175	0.00402	0.06734	0.14718	0.24155	0.34648	0.45341	0.54317	0.58098
0.5	−0.12177	−0.11290	−0.08644	−0.04278	0.01729	0.09234	0.17975	0.27468	0.36770	0.44136	0.47065
0.6	−0.17344	−0.16501	−0.13988	−0.09857	−0.04209	0.02780	0.10800	0.19312	0.27376	0.33485	0.35827
0.7	−0.23317	−0.22520	−0.20150	−0.16270	−0.10997	−0.04538	0.02768	0.10361	0.17351	0.22476	0.24391
0.8	−0.30039	−0.29291	−0.27070	−0.23447	−0.18556	−0.12621	−0.05999	0.00756	0.06830	0.11171	0.12766
0.9	−0.37456	−0.36758	−0.34686	−0.31321	−0.26806	−0.21377	−0.15396	−0.09393	−0.04099	−0.00388	0.00958
1.0	−0.45514	−0.44864	−0.42941	−0.39829	−0.35678	−0.30729	−0.25336	−0.20001	−0.15371	−0.12174	−0.11025
1.1	−0.54159	−0.53557	−0.51779	−0.48910	−0.45105	−0.40604	−0.35749	−0.31003	−0.26938	−0.24165	−0.23175
1.2	−0.63343	−0.62787	−0.61147	−0.58510	−0.55031	−0.50943	−0.46574	−0.42347	−0.38766	−0.36346	−0.35487
1.3	−0.73020	−0.72507	−0.70998	−0.68580	−0.65404	−0.61695	−0.57762	−0.53992	−0.50827	−0.48703	−0.47954
1.4	−0.83145	−0.82674	−0.81289	−0.79074	−0.76178	−0.72816	−0.69274	−0.65905	−0.63098	−0.61227	−0.60569
1.5	−0.93682	−0.93249	−0.91979	−0.89953	−0.87315	−0.84267	−0.81076	−0.78059	−0.75563	−0.73907	−0.73327
1.6	−1.04594	−1.04197	−1.03033	−1.01182	−0.98779	−0.96016	−0.93138	−0.90433	−0.88206	−0.86736	−0.86221
1.7	−1.15848	−1.15485	−1.14420	−1.12729	−1.10541	−1.08035	−1.05437	−1.03007	−1.01016	−0.99705	−0.99248
1.8	−1.27417	−1.27084	−1.26110	−1.24566	−1.22574	−1.20300	−1.17952	−1.15765	−1.13980	−1.12809	−1.12401
1.9	−1.39274	−1.38970	−1.38078	−1.36669	−1.34854	−1.32790	−1.30665	−1.28694	−1.27090	−1.26041	−1.25676
2.0	−1.51396	−1.51117	−1.50302	−1.49015	−1.47361	−1.45485	−1.43562	−1.41782	−1.40338	−1.39395	−1.39068
2.1	−1.63761	−1.63506	−1.62760	−1.61585	−1.60078	−1.58372	−1.56627	−1.55018	−1.53715	−1.52867	−1.52572
2.2	−1.76351	−1.76117	−1.75436	−1.74361	−1.72988	−1.71435	−1.69851	−1.68393	−1.67216	−1.66450	−1.66185
2.3	−1.89149	−1.88934	−1.88311	−1.87329	−1.86076	−1.84662	−1.83221	−1.81899	−1.80834	−1.80142	−1.79902
2.4	−2.02139	−2.01943	−2.01372	−2.00474	−1.99329	−1.98040	−1.96730	−1.95529	−1.94563	−1.93936	−1.93719
2.5	−2.15308	−2.15129	−2.14606	−2.13785	−2.12737	−2.11562	−2.10368	−2.09276	−2.08399	−2.07830	−2.07633

TABLE II

Values of $p = f(q, \nu)$ for the second "stable" region (area between be_1 and bo_2 of Reference 3, p. xliv).

q	ν										
	1.0	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
0.1	1.09873	1.23240	1.45117	1.69718	1.96518	2.25398	2.56319	2.89264	3.24222	3.61190	3.99917
0.2	1.19487	1.28792	1.48255	1.71802	1.98038	2.26574	2.57264	2.90044	3.24879	3.61748	3.99667
0.3	1.28832	1.35923	1.52927	1.75063	2.00468	2.28471	2.58796	2.91310	3.25942	3.62634	3.99250
0.4	1.37899	1.43669	1.58615	1.79255	2.03672	2.31006	2.60855	2.93014	3.27362	3.63785	3.98668
0.5	1.46677	1.51590	1.64907	1.84127	2.07498	2.34076	2.63366	2.95092	3.29079	3.65118	3.97919
0.6	1.55157	1.59475	1.71514	1.89457	2.11792	2.37572	2.66243	2.97472	3.31020	3.66538	3.97005
0.7	1.63329	1.67213	1.78239	1.95063	2.16411	2.41384	2.69398	3.00075	3.33106	3.67941	3.95925
0.8	1.71185	1.74738	1.84949	2.00801	2.21231	2.45407	2.72743	3.02821	3.35255	3.69229	3.94681
0.9	1.78715	1.82007	1.91552	2.06559	2.26143	2.49547	2.76194	3.05631	3.37386	3.70312	3.93274
1.0	1.85911	1.88991	1.97980	2.12251	2.31058	2.53718	2.79670	3.08430	3.39422	3.71118	3.91702
1.1	1.92763	1.95668	2.04185	2.17808	2.35900	2.57847	2.83102	3.11149	3.41293	3.71598	3.89969
1.2	1.99265	2.02021	2.10131	2.23176	2.40610	2.61870	2.86426	3.13724	3.42940	3.71722	3.88074
1.3	2.05408	2.08036	2.15789	2.28314	2.45134	2.65733	2.89587	3.16102	3.44312	3.71480	3.86018
1.4	2.11186	2.13701	2.21136	2.33187	2.49431	2.69388	2.92537	3.18235	3.45373	3.70876	3.83803
1.5	2.16594	2.19008	2.26154	2.37768	2.53467	2.72797	2.95235	3.20086	3.46095	3.69922	3.81429
1.6	2.21626	2.23948	2.30829	2.42035	2.57213	2.75929	2.97649	3.21625	3.46462	3.68636	3.78898
1.7	2.26277	2.28514	2.35150	2.45970	2.60647	2.78757	2.99753	3.22829	3.46467	3.67039	3.76210
1.8	2.30545	2.32702	2.39107	2.49560	2.63751	2.81261	3.01526	3.23684	3.46110	3.65150	3.73368
1.9	2.34426	2.36509	2.42694	2.52795	2.66510	2.83425	3.02954	3.24181	3.45396	3.62989	3.70372
2.0	2.37920	2.39932	2.45906	2.55665	2.68916	2.85238	3.04027	3.24316	3.44335	3.60572	3.67223
2.1	2.41026	2.42969	2.48740	2.58166	2.70958	2.86692	3.04738	3.24090	3.42941	3.57916	3.63924
2.2	2.43744	2.45621	2.51194	2.60295	2.72635	2.87782	3.05088	3.23510	3.41226	3.55034	3.60475
2.3	2.46076	2.47888	2.53269	2.62050	2.73943	2.88508	3.05077	3.22581	3.39206	3.51939	3.56878
2.4	2.48025	2.49774	2.54965	2.63432	2.74884	2.88871	3.04710	3.21314	3.36895	3.48640	3.53135
2.5	2.49593	2.51280	2.56286	2.64443	2.75459	2.88875	3.03994	3.19721	3.34309	3.45147	3.49247

TABLE III

Values of $p = f(q, \nu)$ for the third "stable" region (area between be_2 and bo_3 of Reference 3, p. xliv).

q	ν										
	2.0	2.1	2.2	2.3	2.4	2.5	2.6	2.7	2.8	2.9	3.0
0.1	4.00416	4.41147	4.84130	5.29117	5.76105	6.25095	6.76087	7.29080	7.84073	8.41067	9.00061
0.2	4.01658	4.41595	4.84524	5.29468	5.76421	6.25382	6.76348	7.29318	7.84293	8.41270	9.00238
0.3	4.03706	4.42364	4.85188	5.30057	5.76950	6.25860	6.76783	7.29717	7.84659	8.41608	9.00521
0.4	4.06530	4.43486	4.86135	5.30891	5.77695	6.26533	6.77395	7.30276	7.85172	8.42082	9.00902
0.5	4.10090	4.45000	4.87379	5.31976	5.78662	6.27403	6.78185	7.30997	7.85834	8.42692	9.01372
0.6	4.14338	4.46955	4.88940	5.33322	5.79854	6.28474	6.79155	7.31882	7.86646	8.43439	9.01923
0.7	4.19221	4.49400	4.90837	5.34942	5.81280	6.29749	6.80307	7.32932	7.87607	8.44324	9.02546
0.8	4.24685	4.52378	4.93094	5.36844	5.82945	6.31233	6.81646	7.34148	7.88720	8.45345	9.03234
0.9	4.30673	4.55921	4.95730	5.39042	5.84856	6.32931	6.83173	7.35534	7.89986	8.46506	9.03979
1.0	4.37130	4.60046	4.98763	5.41546	5.87020	6.34845	6.84890	7.37089	7.91404	8.47803	9.04774
1.1	4.44002	4.64748	5.02206	5.44365	5.89443	6.36980	6.86800	7.38816	7.92976	8.49237	9.05612
1.2	4.51239	4.70007	5.06067	5.47507	5.92129	6.39339	6.88906	7.40715	7.94700	8.50805	9.06485
1.3	4.58793	4.75786	5.10348	5.50977	5.95083	6.41925	6.91207	7.42787	7.96578	8.52506	9.07389
1.4	4.66619	4.82037	5.15042	5.54775	5.98307	6.44738	6.93705	7.45031	7.98607	8.54334	9.08316
1.5	4.74678	4.88709	5.20135	5.58899	6.01799	6.47778	6.96400	7.47446	8.00784	8.56286	9.09260
1.6	4.82932	4.95747	5.25609	5.63344	6.05559	6.51045	6.99288	7.50029	8.03107	8.58356	9.10218
1.7	4.91347	5.03099	5.31436	5.68098	6.09580	6.54534	7.02369	7.52778	8.05572	8.60534	9.11182
1.8	4.99893	5.10715	5.37590	5.73150	6.13856	6.58241	7.05637	7.55689	8.08173	8.62814	9.12147
1.9	5.08541	5.18549	5.44036	5.78481	6.18378	6.62161	7.09087	7.58756	8.10903	8.65183	9.13109
2.0	5.17267	5.26560	5.50742	5.84072	6.23132	6.66283	7.12712	7.61972	8.13756	8.67630	9.14063
2.1	5.26046	5.34710	5.57675	5.89902	6.28106	6.70598	7.16506	7.65331	8.16722	8.70141	9.15005
2.2	5.34857	5.42967	5.64800	5.95947	6.33284	6.75096	7.20456	7.68822	8.19790	8.72703	9.15931
2.3	5.43682	5.51302	5.72087	6.02183	6.38649	6.79763	7.24555	7.72435	8.22951	8.75298	9.16836
2.4	5.52503	5.59685	5.79503	6.08586	6.44182	6.84585	7.28787	7.76160	8.26191	8.77911	9.17717
2.5	5.61304	5.68096	5.87022	6.15130	6.49865	6.89546	7.33144	7.79985	8.29497	8.80525	9.18571

No absolute convergence proof was found for the iteration process itself; on the other hand, no convergence instabilities were experienced in the actual computation for the range of parameters that was tabulated.

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An Algorithm for Solving Certain Polynomial Equations with Coefficient Parameters

By Robert D. Stalley

1. Introduction and General Method. We describe a storage-saving procedure that can be used in real time simulation or other situations in which the roots of an equation of the form

$$(1)\qquad \sum_{j=0}^{n} F_j(x_1, x_2, \cdots, x_m)y^{e_j} = 0, \quad m \geqq 2, \quad 0 \leqq e_0 \leqq e_1 \leqq \cdots \leqq e_n,$$

$$e_j \text{ integral } (j = 0, 1, \cdots, n),$$

must be obtained within severe time limits, i.e., from storage, for values of the coefficient parameter m-tuple $(x_1, x_2, \cdots, x_m)$ from some given set. The exponents e_j are defined so as to be monotonically increasing rather than strictly increasing for later notational convenience.

Suppose there exist relations

$$(2)\qquad u_i = \varphi_i(x_1, x_2, \cdots, x_m), \quad (i = \mu, \mu + 1, \cdots, m), \quad 2 \leqq \mu \leqq m,$$

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which give the reduction

$$F_j(x_1, x_2, \cdots, x_m)$$

$$= g(x_1, x_2, \cdots, x_m) f_j(u_\mu, u_{\mu+1}, \cdots, u_m) h^{e_j}(x_1, x_2, \cdots, x_m), \tag{3}$$

$$(j = 0, 1, \cdots, n).$$

Then, if we substitute from (3) into (1), setting

$$yh(x_1, x_2, \cdots, x_m) = v, \tag{4}$$

and divide out $g(x_1, x_2, \cdots, x_m)$, we obtain

$$\sum_{j=0}^{n} f_j(u_\mu, u_{\mu+1}, \cdots, u_m) v^{e_j} = 0. \tag{5}$$

Now, if we store the roots of equation (5) for all values of the $(m - \mu + 1)$-tuple $(u_\mu, u_{\mu+1}, \cdots, u_m)$ within its range, we can find the roots of equation (1) for a given value of the m-tuple $(x_1, x_2, \cdots, x_m)$ within its domain of substitution by computing $(u_\mu, u_{\mu+1}, \cdots, u_m)$ from (2), obtaining from storage the roots of (5), and computing the roots of (1) from (4). Obvious separate considerations must apply if the value of $h(x_1, x_2, \cdots, x_m)$ or $g(x_1, x_2, \cdots, x_m)$ is zero or if any of the above functions are undefined for the given value of $(x_1, x_2, \cdots, x_m)$.

Basically we have replaced m-parameter storage by $(m - \mu + 1)$-parameter storage and a computation which is ordinarily relatively simple. Rather than storing solutions point-wise in m-dimensional space, we store them by $(\mu - 1)$-dimensional manifolds, namely those given by equations (2) taken simultaneously. A side calculation, using equation (4), is needed to obtain the roots for a particular point of the manifold.

This reduction is especially interesting when $\mu = m$, for then we need only single parameter storage. Note that if $m = 2$ we must have $\mu = m$.

We consider two equations for which this reduction can be made. In both equations $\mu = 2$.

2. An Equation of Power-Type. Let equation (1) be

$$\sum_{j=0}^{n} a_j \left(\prod_{i=1}^{m} x_i^{\alpha_i} \right) y^{e_j} = 0, \quad a_j \neq 0, \quad (j = 0, 1, \cdots, n), \tag{6}$$

where

$$\alpha_1 + \sum_{i=2}^{m} r_i \alpha_i = k + r e_j, \quad (r_2, r_3, \cdots, r_m, r, k \text{ constant}; \tag{7}$$

$$\alpha_1, \alpha_2, \cdots, \alpha_m \text{ constant for each value of } j).$$

To reduce equation (6), we first use (7) to obtain

$$\prod_{i=1}^{m} x_i^{\alpha_i} = x_1^k x_1^{re_j} \prod_{i=2}^{m} x_1^{-r_i \alpha_i} \prod_{i=2}^{m} x_i^{\alpha_i} = x_1^k \left\{ \prod_{i=2}^{m} (x_1^{-r_i} x_i)^{\alpha_i} \right\} x_1^{re_j}.$$

Then, after dividing out x_1^k, equation (6) becomes

$$\sum_{j=0}^{n} a_j \left(\prod_{i=2}^{m} u_i^{\alpha_{ij}} \right) v^{e_j} = 0,$$

where

$$u_i = x_1^{-r_i} x_i, \qquad (i = 2, 3, \cdots, m),$$

and

$$v = x_1^r y.$$

This completes the reduction.

Homogeneous polynomial equations, with y as one of the variables, are special cases where $r_2 = r_3 = \cdots = r_m = 1$, $r = -1$, and α_i is a non-negative integer, $(i = 1, 2, \cdots, m)$.

Now we look at the case $m = 2$. Set $x_1 = w$, $x_2 = z$, $\alpha_1 = \alpha$, $\alpha_2 = \beta$, $r_2 = s$, $r = t$, and $u_2 = b$. The equation

$$\sum_{j=0}^{n} a_j w^{\alpha} z^{\beta} y^{e_j} = 0, \qquad (\alpha + s\beta = k + te_j), \tag{8}$$

is reduced to the equation

$$\sum_{j=0}^{n} a_j b^{\beta} v^{e_j} = 0, \tag{9}$$

where

$$b = w^{-s} z \tag{10}$$

and

$$v = w^t y. \tag{11}$$

Thus, if the roots of (9) are stored for all of the values of b in its range, we have the following algorithm for solving equation (8) for given values of w and z:

I. Compute w^s and w^t;
II. Compute b from (10);
III. Look up the roots of (9);
IV. Compute y from (11).

The algorithm is especially simple if s and t are integers. We have combined storage lookup and computation to obtain an algorithm for solving (8) which involves little computation and only single parameter storage. The writer once needed a real time series solution of an equation of the form of (8) where $e_n = 6$, $s = \frac{2}{3}$, $t = -\frac{1}{3}$, and where, of course, the remaining arbitrary constants were equal to specified values [1].

This equation of power type, equation (6) with condition (7), has many generalizations which are reducible in the same way. The functions F_j may be quotients of polynomials whose terms are of the form of the coefficients of equation (6), where e_j is constant for all terms of F_j, where k and r are constant for each polynomial, and where the values of k and r for any numerator minus the corresponding

values of k and r for the corresponding denominator are constant. Moreover, the terms may contain other factors and the parameters may themselves be functions of other parameters.

3. An Equation of Exponential Type. Let equation (1) be

$$\sum_{j=0}^{n} a_j \left(\prod_{i=1}^{m} \alpha_i^{x_i} \right) y^{e_j} = 0, \qquad a_j \neq 0, \qquad (j = 0, 1, \cdots, n), \tag{12}$$

where

$$\alpha_1 \prod_{i=2}^{m} \alpha_i^{r_i} = k c^{e_j}, \qquad (r_2, r_3, \cdots, r_m, c, k \text{ constant}; \tag{13}$$

$$\alpha_1, \alpha_2, \cdots, \alpha_m \text{ constant for each value of } j), \quad c \neq 0, k \neq 0.$$

To reduce equation (12), we first use (13) to obtain

$$\prod_{i=1}^{m} \alpha_i^{x_i} = k^{x_1} c^{x_1 e_j} \prod_{i=2}^{m} \alpha_i^{-r_i x_1} \prod_{i=2}^{m} \alpha_i^{x_i} = k^{x_1} \left(\prod_{i=2}^{m} \alpha_i^{-r_i x_1 + x_i} \right) c^{x_1 e_j}.$$

Then, after dividing out k^{x_1}, equation (12) becomes

$$\sum_{j=1}^{n} a_j \left(\prod_{i=2}^{m} \alpha_i^{u_i} \right) v^{e_j} = 0,$$

where

$$u_i = x_i - r_i x_1, \qquad (i = 2, 3, \cdots, m),$$

and

$$v = c^{x_1} y.$$

This completes the reduction.

There are specializations and generalizations of this equation of exponential type analogous to those for the equation of power type.

4. Remarks. The restriction that e_j be integral, $(j = 0, 1, \cdots, n)$, is not used and hence may be removed throughout this paper. With the customary interpretation of notation and statements, we may extend this paper to cover the trivial case $\mu = m + 1$, and consequently the trivial case $m = 1$.

An open problem is the enumeration of all examples, from various specified classes, for which the method of this paper is applicable. Another open problem, perhaps more interesting, is that of solving approximately polynomial equations with coefficient parameters by finding and solving approximating polynomial equations of a type for which the method of this paper is applicable. The generality of equation (6) with condition (7) and equation (12) with condition (13) make them promising approximating polynomial equations.

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1. R. D. STALLEY, *Solution of a Polynomial Equation*, Remington Rand, Inc., Engineering Research Associates Division, St. Paul, 1955, limited distribution.

A Remark Concerning the Solution of the Dirichlet Problem by Finite Differences

By Bernard Epstein

In the classic paper [1] it is proven that the mesh-functions obtained by solving a discrete analogue of the Dirichlet problem converge, as the mesh-width approaches zero, to a harmonic function which solves the Dirichlet problem in a somewhat generalized sense. More precisely, let the function f be continuously differentiable in a bounded plane domain G and continuous in the closure $\bar{G}$ of G, and let the Dirichlet integral

$$D(f) = \iint_G [f_x^2 + f_y^2]\, dx\, dy \tag{1}$$

be finite. Then the finite-difference method presented in the aforementioned paper is shown (under suitable assumptions concerning the smoothness of the boundary) to furnish a function u harmonic in G whose Dirichlet integral $D(u)$ is finite (and not greater than $D(f)$), and it is shown that u agrees with f on the boundary ∂G in the sense that for all sufficiently small values of ϵ the inequality

$$\iint_{S_\epsilon} (u - f)^2\, dx\, dy \leqq \gamma\epsilon^2 \tag{2}$$

holds, where S_ϵ denotes the portion of G consisting of those points whose distance from ∂G is less than ϵ and γ denotes some positive number independent of ϵ. The authors indicate, without supplying the details, that u agrees with f on ∂G in the more elementary sense that the function $u - f$, initially defined only in G, becomes continuous in $\bar{G}$ if defined to vanish everywhere on ∂G. In this brief note we present a proof of this fact, thus showing, without reference to any other method of treating the Dirichlet problem, that the finite-difference method provides an existence proof for the "conventional" formulation of the Dirichlet problem as well as an effective procedure for computing the solution.

The proof is accomplished by establishing two lemmas. The first of these is a strengthened version of the inequality (2).

$$\text{LEMMA 1. } \lim_{\epsilon \to 0} \epsilon^{-2} \iint_{S_\epsilon} (u - f)^2\, dx\, dy = 0. \tag{3}$$

Proof. First we establish (3) in the particular case that G is the unit disc, $x^2 + y^2 < 1$. Let $v = u - f$ and let two points with polar coordinates (r, θ), (R, θ), $r < R < 1$, be selected. Then

$$v(R, \theta) - v(r, \theta) = \int_r^R v_\rho(\rho, \theta)\, d\rho = \int_r^R v_\rho(\rho, \theta)\rho^{1/2} \cdot \rho^{-1/2}\, d\rho. \tag{4}$$

Applying the Schwarz inequality and assuming that $r > \frac{1}{2}$, we obtain

$$[v(R, \theta) - v(r, \theta)]^2 \leqq \left(\int_r^R v_\rho^2 \rho\, d\rho\right)\left(\int_r^R \rho^{-1}\, d\rho\right) \leqq 2(1 - r) \int_r^1 v_\rho^2 \rho\, d\rho. \tag{5}$$

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Integrating over θ and setting $\epsilon = 1 - r$, we obtain

$$(6) \qquad \int_0^{2\pi} [v(R,\theta) - v(r,\theta)]^2 \, d\theta \leqq 2\epsilon \iint_{S_\epsilon} v_\rho^2 \, dx\, dy \leqq 2\epsilon \iint_{S_\epsilon} [v_x^2 + v_y^2] \, dx\, dy \leqq 2\epsilon D(v).$$

Since $D(f)$ and $D(u)$ exist, $D(v)$ also exists, so the left side of (6) must approach zero with ϵ. It then follows, by a familiar argument, that $\lim_{\epsilon \to 0} \int_0^{2\pi} v^2(r, \theta) \, d\theta$ exists. If this limit, which we denote by C, were positive, the inequality

$$\int_0^{2\pi} v^2(r, \theta) \, d\theta > \tfrac{1}{2}C$$

would hold for all sufficiently small ϵ. Integrating this inequality, we would obtain, in contradiction with (2),

$$(7) \qquad \iint_{S_\epsilon} v^2(r,\theta) \, r dr \, d\theta > \frac{1}{2} C \int_{1-\epsilon}^1 r dr = \frac{1}{4} C\epsilon(2 - \epsilon).$$

Having thus established that $C = 0$, we return to (6), let $R \to 1$ (keeping r momentarily fixed), and thus obtain, by an obvious application of the triangle inequality,

$$(8) \qquad \int_0^{2\pi} v^2(r, \theta) \, d\theta \leqq 2\epsilon D_\epsilon(v)$$

where $D_\epsilon(v) = \iint_{S_\epsilon} [v_x^2 + v_y^2] \, dx\, dy$. Integrating and noting that $D_\epsilon(v)$ is a decreasing function of $r(= 1 - \epsilon)$, we obtain

$$(9) \qquad \iint_{S_\epsilon} v^2 \, dx\, dy \leqq 2\epsilon D_\epsilon(v) \int_{1-\epsilon}^1 \rho \, d\rho = \epsilon^2(2 - \epsilon) D_\epsilon(v).$$

Since $D_\epsilon(v)$ approaches zero with ϵ, (9) implies (3).

For a more general domain with sufficiently smooth boundary it is readily seen that the same argument may be applied by introducing a coordinate system whose coordinate curves are the normals to the boundary and their orthogonal trajectories.

Before stating the second lemma, we introduce the following terminology (cf. [2, p. 481]). A $\dot{D}$-function (defined in a domain G) is one that is continuously differentiable in G and has compact support, i.e., there exists $\epsilon > 0$ such that the function vanishes throughout S_ϵ. A $\mathring{D}$-function, say v, is one which is continuously differentiable in G and can be approximated by $\dot{D}$-functions in the sense that there exists a sequence $\{v_n\}$ of $\dot{D}$-functions such that

$$(10) \qquad \lim_{n\to\infty} \iint_G (v - v_n)^2 \, dx\, dy = 0, \lim_{n\to\infty} D(v - v_n) = 0.$$

LEMMA 2. *$v(= u - f)$ is a $\mathring{D}$-function.*

Proof. As in the proof of the previous lemma, it suffices to confine attention to the case that G is the unit disc. For each positive integer n we define the func-

tions g_n and h_n as follows:

$$(11)\quad g_n = \begin{cases} 1 & , \quad 0 \leqq r \leqq 1 - \dfrac{1}{n} \\ \cos^2 n\pi\left(1 - r - \dfrac{1}{n}\right) & , \quad 1 - \dfrac{1}{n} \leqq r \leqq 1 - \dfrac{1}{2n} \\ 0 & , \quad r \geqq 1 - \dfrac{1}{2n} \end{cases}, \quad h_n = 1 - g_n .$$

Then the functions $v_n = vg_n$ are obviously $\mathring{D}$-functions, and we establish the two parts of (10) by the following arguments.

(a)
$$\iint_G (v - v_n)^2\, dx\, dy = \iint_G (vh_n)^2\, dx\, dy$$
$$= \iint_{S_{1/n}} (vh_n)^2\, dx\, dy \leqq \iint_{S_{1/n}} v^2\, dx\, dy \to 0;$$

(b)
$$D(v - v_n) = D(vh_n) = D_{1/n}(vh_n) \leqq D_{1/n}(vh_n)$$
$$+ \iint_{S_{1/n}} [v_x h_n - vh_{n,x})^2 + (v_y h_n - vh_{n,y})^2]\, dx\, dy$$
$$= 2 \iint_{S_{1/n}} [h_n^2(v_x^2 + v_y^2) + v^2(h_{n,x}^2 + h_{n,y}^2)]\, dx\, dy.$$

Since $h_n^2 \leqq 1$ and $h_{n,x}^2 + h_{n,y}^2 \leqq n^2\pi^2$, we obtain

$$D(v - v_n) \leqq 2D_{1/n}(v) + 2n^2\pi^2 \iint_{S_{1/n}} v^2\, dx\, dy.$$

$D_{1/n}(v)$ approaches zero with increasing n, and the same is true of the remaining term, by Lemma 1. Thus the present lemma is proven.

The desired result concerning the boundary behavior of u now follows immediately from the following theorem (which is a particular case of a theorem proven in [2, p. 495–7]): Let u be harmonic in a bounded domain G, let f be continuous in $\bar{G}$, let $D(u)$ and $D(f)$ be finite, and let $u - f$ be a $\mathring{D}$-function. Then $u - f$ approaches zero at the boundary if the latter satisfies certain mild conditions.

Petrovsky [3] has presented a proof that the boundary values are assumed. His proof, which also appears in [4, p. 186 ff.], does not make use of (2) at all. However, the Petrovsky proof does not appear to extend readily to more general elliptic equations, whereas the proof of (2) presented in [1] and the argument presented here can be suitably modified so as to apply to other equations.

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1. R. Courant, K. Friedrichs & H. Levy, "Über die partiellen Differenzengleichungen der mathematischen Physik," *Math. Ann.*, v. 100, 1928, p. 32–74.
2. R. Courant & D. Hilbert, *Methoden der mathematischen Physik*, v. 2, J. Springer, Berlin, 1937.
3. I. G. Petrovsky, "New proof of the existence of a solution of Dirichlet's problem by the method of finite differences," *Uspehi Mat. Nauk.*, v. 8, 1941, p. 161–170.
4. J. D. Tamarkin & W. Feller, *Partial Differential Equations*, Mimeographed lecture notes, Brown University, Providence, 1941.

REVIEWS AND DESCRIPTIONS OF TABLES AND BOOKS

1[C, K, L]. GERALD J. LIEBERMAN & DONALD B. OWEN, *Tables of the Hypergeometric Probability Distribution*, Stanford University Press, California, 1961, 7 + 726 p., 24 cm. Price $15.00.

In this volume there are three main tables of the hypergeometric probability distribution and a table of logarithms of factorials. The nomenclature of sampling inspection is used to describe the parameters of the hypergeometric probability distribution. The main tables give the values of $p(x) = p(N, n, k, x)$ and $P(x) = P(N, n, k, x)$, where

N = number of items in a lot,
n = number of items in a sample taken from the lot,
k = number of defective items in the lot,
x = number of defective items observed in the sample.

Then, the probability

$$p(x) = \Pr\{\text{Exactly } x \text{ defectives are found in sample}\}$$

$$= \frac{k!\,n!}{(k-x)!\,(n-x)!\,x!} \cdot \frac{(N-k)!\,(N-n)!}{N!(N-k-n+x)!},$$

x being an integer such that $[0, n + k - N] \leqq x \leqq \min [n, k]$, and $P(x) = Pr\{x \text{ defectives or fewer are found in sample}\}$

$$= \sum_{i=M}^{x} \frac{k!\,n!}{(k-i)!\,(n-i)!\,i!} \cdot \frac{(N-k)!\,(N-n)!}{N!(N-k-n+i)!},$$

where $M = \max [0, n + k - N]$.

The first table lists the values of $p(x)$ and $P(x)$ to six decimal places for $N = 2(1)49,\ 50(10)100,\ n = 1(1)\ N-1,\ k = 1(1)\ n,\ x = 0(1)\ k$, for $N \leqq 25$. For $N > 25$, the values of $p(x)$ and $P(x)$ are given only up to

$$n = \frac{N}{2} \quad \text{or} \quad n = \frac{N-1}{2},$$

N even or odd, respectively. The authors note that by the use of certain symmetry relationships, all possible $p(x)$ and $P(x)$ can be obtained.

The second table gives the values of $p(x)$ and $P(x)$ to six decimal places for $N = 1000,\ n = 500,\ k = 1(1)\ 500,\ x = 0(1)\ k/2$ (k even), $(k-1)/2$ (k odd). Entries are omitted when $p(x) < 10^{-6}$.

The third table gives the values of $p(x)$ and $P(x)$ to six decimal places for $N = 100(100)\ 2000,\ n = \frac{1}{2}N,\ k = n-1,\ n$, and $x = 0(1)\ n/2$ (n even), $(n-1)/2$ (n odd). Entries are omitted when $p(x) < 10^{-6}$.

The fourth table is a list of $\log N!$ for $N = 1(1)\ 2000$ taken from *Logarithms of Factorials from 1 to 2000*, by D. B. Owen and C. M. Williams, Sandia Corporation Monograph SCR-158, December 1959. Values of $\log N!$ in this table are given to fifteen decimal places, and were used for the calculation of $p(x)$ and $P(x)$. The values of $p(x)$ and $P(x)$ in all tables are claimed to have been computed correct to at least eight decimal places before they were rounded to six decimal places.

The introductory part of this volume includes the definitions of the hypergeometric function and the various symmetry relationships, applications, approximations and interpolations, a summary of some useful formulas on sums of combinatorials, and a bibliography of 66 references. Examples given in applications include sequential procedure, test of the equality of two proportions, distribution of the number of exceedances, Bayesian prediction, and sampling inspection.

The reviewer's immediate reaction to these tables is that the type face is too small for easy reading and that the format makes it difficult to find the values of the indexing parameters. However, considering the 135,874 entries and the 726 pages, it would be difficult to eliminate these faults without prohibitive increase in both the size and cost of this volume.

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2[G, I, X, Z]. Ralph G. Stanton, *Numerical Methods for Science and Engineering*, Prentice-Hall, Inc., Englewood Cliffs, N. J., 1961, xii + 266 p., 23 cm. Price $9.00.

This book is designed as a textbook for an introductory course in numerical methods for students in the physical sciences and engineering with a good knowledge of calculus and differential equations. The selection of topics is fairly standard, as one would gather from the following chapter headings: Ordinary Finite Differences, Divided Differences, Central Differences, Inverse Interpolation and the Solution of Equations, Computation with Series and Integrals, Numerical Solution of Differential Equations, Linear Systems and Matrices, Solution of Linear Equations, Difference Equations, Solution of Differential Equations by Difference Equation Methods, and the Principles of Automatic Computation.

The author states that the book was developed from the standpoint of hand and desk-calculator techniques, and justifies this on the grounds of his belief that "the majority of workers in science and engineering can make great use of numerical methods without perhaps ever encountering a problem of sufficient length or complexity to justify programming it for an electronic computer." His final chapter, containing only eighteen pages about automatic computation, seems to confirm one's belief that the author views the modern field of numerical computation with automatic electronic computers as a spectator rather than as a participant.

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3[G, S]. Taro Shimpuku, "General Theory and Numerical Tables of Clebsch-Gordan Coefficients," *Progr. Theoret. Phys.*, Kyoto, Japan, Supplement No. 13, 1960, p. 1–135.

General formulas for the Clebsch-Gordan coefficients $(j_1 j_2 m_1 m_2 \mid j_1 j_2 j m)$, in the notation of Condon and Shortley [1], have been given by Wigner and by Racah [2], [3]. These formulas are very complex and computationally inconvenient. Shimpuku states: "Here we derive a new general expression of $C - G$ coefficients

from the theory of spinor representation in three-dimensional rotation group, and this expression has a convenient form for practical evaluation (for any given values of the parameters)."

Algebraic formulas for these coefficients, for the special cases $j_2 = \frac{1}{2}, 1, \frac{3}{2}, 2$ are given in [1], p. 76–77; similar formulas for $j_2 = \frac{5}{2}$ and 3 are available in sources noted in the references in Shimpuku's paper. Shimpuku tabulates the algebraic formulas for $j_2 = \frac{7}{2}, 4, \frac{9}{2}$, and 5.

Numerical tables have been compiled, by Simon at Oak Ridge, for all cases where $j_1 \leqq \frac{9}{2}$, $j_2 \leqq \frac{9}{2}$. Over 100 pages of numerical tables are given by Shimpuku; these tables cover $j_2 = 5, \frac{11}{2}$, and 6 for all $j_1 \leqq 6$. Each entry is expressed as the radical of a rational fraction.

Shimpuku does not refer to the recent tabulation by Rotenberg et al. [4] of $3 - j$ symbols, from which the Clebsch-Gordan coefficients can be readily obtained. This tabulation covers all values $j_1 \leqq 8$, $j_2 \leqq 8$. However, for the range covered by Shimpuku, many users may find his rational fractions more convenient than the expressions as products of powers of primes used by Rotenberg.

GEORGE SHORTLEY

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1. E. U. CONDON & G. H. SHORTLEY, *The Theory of Atomic Spectra*, Cambridge University Press, New York, 1935, p. 75.
2. E. P. WIGNER, *Gruppentheorie und ihre Anwendung auf die Quantenmechanik der Atomspektren*, Friedrich Vieweg und Sohn, Braunschweig, 1931.
3. G. RACAH, "Theory of complex spectra II," *Phys. Rev.*, v. 62, 1942, p. 438.
4. M. ROTENBERG, R. BIVINS, N. METROPOLIS & J. K. WOOTEN, JR., *The 3-j and 6-j Symbols*, Technology Press, Cambridge, 1960. See *Math. Comp.*, v. 14, 1960, p. 382–383, Review 71.

✓ **4[I, X, Z].** NATIONAL PHYSICAL LABORATORY, *Modern Computing Methods*, Second Edition, Her Majesty's Stationery Office, London, 1961, 25 cm. Price $3.78.

This is the second edition of a booklet I praised highly when I reviewed its first edition (*MTAC*, v. 12, 1958, p. 230, Review 96). However, much has changed since then, and while I still feel that I shall recommend to every budding numerical analyst that he consult this booklet, I must add here to its list of limitations. I am tempted to say that it is "modern," much the same as Gilbert and Sullivan's Major General, but this would be entirely too harsh.

The booklet contains nothing about linear programming, assignment problems, or discrete variable calculations, which play a large role in computation, at least in the United States. (Beale in the United Kingdom might claim that these problems occur there also.) There is nothing about the Monte Carlo method which is very popular, at least in the southwest sections of the United States. (Hammersley in the United Kingdom might claim that these problems occur there also.) There is essentially nothing (nine lines of text, washing their hands of the whole subject) concerning latent roots and characteristic vectors of unsymmetric matrices, although some of these problems are vital in the study of stability. (This is most disappointing of all, for the workers at the National Physical Laboratory were spectacular in their early attacks on matrix problems and their reporting of their experiences.) There is a tendency to make overly dogmatic statements: "For

hyperbolic equations the existence of real and distinct characteristics leads to the most satisfactory known method of numerical solution" (p. 105).

The listing of tables of functions of several variables is not adequately described. In particular, the alluring paper by Kolmogoroff [1] is not mentioned. In general, scant attention is paid to important Russian work; the book by Kantorovich and Krylov [2] is not listed in the bibliography, even though it is available in an understandable English translation.

Despite these criticisms, which might be likened to the disappointment of a lover (of the first edition) as his love ages, this is a handy booklet to have available. It is a cook book of procedures which are recommended on the experience of a perceptive, scholarly, and active computing group. It is less necessary now than it was when it was published in its first edition, for SHARE and the other users' groups have made experiences with computers more easily available to other users, but this booklet is more precise, less coding-conscious, and more scholarly than the reports of the users' groups. The booklet has been brought up to date on the topics it covers; Givens and Householder on latent roots are quoted carefully, including a British interpretation of their impressive work in both avoidance of long calculation and analysis of error. On the other hand, many reports of computational experience now exist in the literature, which was not the case when the first edition was published, so the booklet is no longer a must.

I would feel unhappy if I knew of this volume and did not have it in my library. I suggest that firms which have spent millions of dollars on computers buy a few copies even though some isomorph of SHARE is available.

If a third edition is contemplated, I suggest that the chapter on Finite Difference Methods be omitted as non-modern. By implication above, I have suggested chapters which should be included. Also a chapter on coding and coding languages might reasonably be added.

I note that there is considerable modernization of outlook (including a chapter on Chebyshev series), and this is good.

C. B. Tompkins

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1. A. N. Kolmogorov, "O predstavlenii nepreryrnykh funktsiĭ neskopkikh peremennykh v vide superpozitsii neprerývnykh funktsiĭ odnogo peremennogo i slozheniia" ("On the representation of a continuous function of several variables in the form of a superposition of continuous functions of one variable and their sums"), Akad. Nauk SSSR, *Doklady*, 1957, v. 114, p. 953–6.
2. L. V. Kantorovich & V. I. Krylov, *Approximate Methods of Higher Analysis*, translated by Curtis Benster, Noordhoff, Groningen, 1958.

5[K]. G. W. Rosenthal & J. J. Rodden, *Tables of the Integral of the Elliptical Bivariate Normal Distribution over Offset Circles*, LMSD-800619, Lockheed Missiles and Space Division, Sunnyvale, California, May 1961, iii + 92 p., 28 cm.

These tables give the probabilities of being inside various circles not about the mean from a bivariate normal distribution having unequal variances. The range of the tables includes values of the mean up to three times the standard deviation.

Author's Summary

6[K, X]. V. V. SOLODOVNIKOV, *Introduction to the Statistical Dynamics of Automatic Control Systems*, Translation edited by John B. Thomas and Lotfi A. Zadeh, Dover Publications, New York, 1960, xx + 307 p., 20 cm. Price $2.25 (Paperbound).

This book, first published in Russian in 1952, gives an excellently written, self-contained account of the principles of the analysis of linear systems, the statistics of random signals, and the theory of linear prediction and filtering. The translation is well done. In addition to treating exact methods, the author discusses methods of obtaining approximate solutions to various problems.

The first three chapters are devoted to a discussion of the theory of the transients in a linear system produced by deterministic signals, to the elements of probability theory, and to the basic concepts of the theory of stationary random processes.

Chapter IV discusses the criterion of least mean-square error. Linear and square-law detectors are used to show how some nonlinear systems may be treated.

In Chapter V the problem of using numerical methods to approximate spectral distribution curves is treated.

Chapters VI, VII, and VIII contain the derivation and application of formulas from which one may obtain the transfer function yielding a minimum mean-square error from the knowledge of the spectral densities of the signal and noise. The last of these chapters treats the case where the signal is composed of two parts, one deterministic and one random.

The book contains four appendices. Appendix I consists of five-place tables of the functions $\frac{\sin x}{x}$ and $\frac{\cos x}{x}$ for $x = 0(.01)10.0(.1)20(1)100$. Appendix II contains tables of the first five Laguerre functions to five significant figures for values of the argument in the range $0(.01)1.0(.1)20(1)30$. Appendices IIIa and IIIb give five-place tables for the calculation of the so-called phase characteristic function from straight-line approximations of the logarithm of the spectral-density function. Appendix IV gives a table of integrals

$$I_n = \frac{1}{2\pi j}\int_{-\infty}^{\infty} \frac{G_n(jw)}{H_n(jw)H_n(-jw)}\,dw,\ n = 1(1)7,$$

where

$$G_n(jw) = b_0(jw)^n + b_1(jw)^{n-1} + \cdots + b_n,$$

$$H_n(jw) = A_0(jw)^n + A_1(jw)^{n-1} + \cdots + A_n,$$

and all roots of $H_n(jw)$ are in the upper half-plane.

A. H. T.

7[L]. DAVID J. BENDANIEL & WILLIAM E. CARR, *Tables of Solutions of Legendre's Equation for Indices of Nonintegral Order*, University of California Lawrence Radiation Laboratory, Livermore, UCRL-5859, September, 1960, 68 p., 28 cm. Available from the Office of Technical Services, Washington 25, D. C. Price $1.75.

We employ the usual notation for hypergeometric and Legendre functions [1]. Let

$$f_1(x) = {}_2F_1(-\nu/2, \nu/2 + \tfrac{1}{2}; \tfrac{1}{2}; x^2);\quad f_2(x) = x\,{}_2F_1(\tfrac{1}{2} - \nu/2, 1 + \nu/2; 3/2; x^2).$$

Then $f_1(x)$ and $f_2(x)$ satisfy the differential equation

$$(x^2 - 1)\, d^2f/dx^2 + 2x\, df/dx - \nu(\nu + 1)f = 0,$$

and we have

$$P_\nu(x) = a_1 f_1(x) + a_2 f_2(x), \qquad Q_\nu(x) = b_1 f_1(x) + b_2 f_2(x),$$

where

$$a_1 = \pi^{1/2}[\Gamma(\tfrac{1}{2} - \nu/2)\Gamma(1 + \nu/2)]^{-1}, \quad a_2 = -2\pi^{1/2}[\Gamma(\tfrac{1}{2} + \nu/2)\Gamma(-\nu/2)]^{-1}$$

$$b_1 = -\frac{\tfrac{1}{2}\pi^{1/2}\Gamma(\tfrac{1}{2} + \nu/2)\sin \nu\pi/2}{\Gamma(1 + \nu/2)}, \quad b_2 = \frac{\pi^{1/2}\Gamma(1 + \nu/2)\cos \nu\pi/2}{\Gamma(\tfrac{1}{2} + \nu/2)}.$$

Table 1 gives $f_1(x)$ to 5S, corresponding to $x = 0(0.01)0.99$, $\nu = 0(0.0625)1(0.125)10(0.25)36$.

For a given ν, let $\alpha_i^{(\nu)}$ be the i-th zero of $f_1(x)$. Then Table 2 gives 4S values of $\alpha_i^{(\nu)}$ and of the integrals

$$\int_0^{\alpha_i^{(\nu)}} f_1(x)\, dx, \quad \int_0^{\alpha_i^{(\nu)}} \{f_1(x)\}^2\, dx, \quad \text{for} \quad \nu = 0.4375(0.0625)36.$$

Tables 3 and 4 present corresponding data for $f_2(x)$.

The tables are essentially new, and were obtained on an automatic computer. An introduction gives a few definitions (the coefficients a_1 and a_2 in the formula for $P_\nu(x)$ contain typographical errors). There is no discussion of formulas used to perform and check the calculations. No attempt is made to give closed-form results, which would be useful to the applied worker. For example, we can show that

$$f_1(x) = (\cos\theta/2)^{-1/2} \cos\left\{N_1\theta + \frac{1}{16N_1}(\tan\theta/2 + \theta/2)\right\}\{1 + O(1/\nu^2)\},$$

$$\int_0^x f_1(t)\, dt = (2N_1)^{-1}(\cos\theta/2)^{1/2} \sin\left\{N_1\theta + \frac{1}{16N_1}(-3\tan\theta/2 + \theta/2)\right\} \cdot\{1 + O(1/\nu^2)\},$$

$$f_2(x) = (2N_2)^{-1}(\cos\theta/2)^{-1/2} \sin\left\{N_2\theta + \frac{1}{16N_2}(\tan\theta/2 + 9\theta/2)\right\} \cdot\{1 + O(1/\nu^2)\},$$

$$\int_0^x f_2(t)\, dt = -(2N_2)^{-2}(\cos\theta/2)^{1/2} \cos\left\{N_2\theta + \frac{3}{16N_2}(-\tan\theta/2 + 3\theta/2)\right\} \cdot\{1 + O(1/\nu^2)\}$$

$$N_1^2 = \nu(\nu + 1)/4, \quad N_2^2 = (\nu - 1)(\nu + 2)/4, \quad x = \sin\theta/2, \quad 0 \leqq \theta < \pi.$$

Indeed, using these results, we have spot checked the entries. Tables 1 and 3 and values of the zeros of f_1 and f_2 in Tables 2 and 4 appear to be correct.

If $\nu = 20$, and $\alpha = 1, 2, 3$, the values of $\int_0^{x_\alpha} f_1(t)\, dt$ are erroneous, as is also the value of $\int_0^{x_1} f_1^2(t)\, dt$. It is curious that the error in each case is about 0.0021.

Similar check calculations for $f_2(x)$ reveal a persistent error of about 0.6024. Thus the tables of the numerical values of the integrals should be used, if at all, with caution. We have corresponded with one of the authors (D.J.B.). He has checked those entries in Tables 1 and 3 against known values of Legendre polynomials and finds that they are correct. The reason for the bias in the values of the integrals is not known, but he suspects that it arises from the binary-to-decimal conversion. We conclude with the "trite" observation that automatic computers cannot be trusted implicitly, and that the need for analysis and checking remains.

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1. A. Erdélyi, et al., *Higher Transcendental Functions*, Vol. 1, McGraw-Hill, New York, 1953.

8[L]. Ludo K. Frevel & J. W. Turley, *Tables of Iterated Sine-Integral*, The Dow Chemical Company, Midland, Michigan, 1961. Deposited in UMT File.

Three tables of decimal values of the iterated sine-integral, $Si^{n}(x)$, are herein presented, as computed on a Burroughs 220 system supplemented by Cardatron equipment, which permitted on-line printing of the results in the desired tabular format.

Table 1 presents the values of $Si^{n}(x)$ to 9D for $n = 1(1)10$, $x = 0(0.2)10$. Table 2 gives values of this function to 7D for $n = 0(0.05)10$, π, 2π, 3π, and Table 3 gives for $n = 1(1)10$ the values to 9D of the first thirty extrema, which correspond to $x = m\pi$, where $m = 1(1)30$.

In an accompanying text of three pages the authors describe in detail the method of calculation and the underlying mathematical formulas. It is there stated that the entries in Table 2 were computed to 9D prior to rounding. The entries in Table 3 are claimed to be accurate to within a unit in the final decimal place, and the authors imply in their explanatory text that comparable accuracy was attained in the computation of the entries in Table 1.

The tabular data corresponding to the values of n different from unity constitute an original contribution to the literature of mathematical tables.

J. W. W.

9[L, X]. Hans Sagan, *Boundary and Eigenvalue Problems in Mathematical Physics*, John Wiley & Sons, Inc., New York, 1961, xviii + 381 p., 24 cm. Price $9.50.

This attractive newcomer to the ranks of the textbooks on methods of mathematical physics comes to us directly from Moscow (where, for the past four years, the author has been an Associate Professor of Mathematics at the University of Idaho). This book contains material which has been used in the author's classes to seniors and beginning graduate students in mathematics, applied mathematics, physics, and engineering for the past five years. The author's stated purpose is not to present a vast number of seemingly unrelated mathematical techniques and tricks that are used in the mathematical treatment of problems which arise in

physics and engineering, but rather to develop the material from a few basic concepts; namely, Hamilton's principle together with the theory of the first variation, and Bernoulli's separation method for the solution of linear homogeneous partial differential equations. The author's persuasive style appears certain to gain adherents for his viewpoints on many college campuses this coming fall.

Hamilton's principle and the theory of the first variation occupy Chapter 1. The representation of some physical phenomena by partial differential equations (vibrating string and membrane, heat conduction and potential equation) forms the subject matter of Chapter II. Chapter III contains general remarks on the existence and uniqueness of solutions and the presentation of Bernoulli's method of separation of variables, while Chapter IV is devoted to Fourier series. Chapter V deals with self-adjoint boundary-value problems, the concept of their eigenvalues being developed according to the elementary method of H. Pruefer in *Mathematische Annalen*, v. 95 (1926). Chapters VI and VIII, on special functions, deal with Legendre polynomials and Bessel functions, and spherical harmonics, respectively. Chapter VII develops the characterization of eigenvalues by a variational principle; while the final Chapter IX is devoted to the nonhomogeneous boundary-value problem (Green's function and generalized Green's function).

The text is well designed for class room use. The author intends it to be used in a two-semester three-credit course. Each chapter is generously provided with interesting exercises (answers and hints are provided at the end of the book for the even-numbered problems). A recommended supplementary reading list concludes each chapter. A welcome innovation is the detailed appendix, containing a condensation of topics with which "the student who wishes to take this course with a reasonable chance to succeed should be familiar."

J. B. Diaz

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10[S]. Charles DeWitt Coleman, William R. Bozman & William F. Meggers, *Table of Wavenumbers*, Volumes I and II, U. S. Department of Commerce. Volume I—2000 A to 7000 A, and Volume II—7000 A to 1000 μ, 1960, vii + 500 p., and vii + 534 p., 35 cm. Price $6.00.

A two-volume table for converting wave lengths in standard air to wave numbers in vacuum was computed by using the equation $\sigma_{vac} = 1/(n\lambda_{air})$, where n was computed from Edlen's 1953 equation for the refractive index of air. Wave numbers are given to the nearest 0.001 K (cm^{-1}) for wave lengths from 2000 to 7000 A in volume I, and 7000 A to 1000 μ in volume II. Proportional tables are given for linear interpolation between entries of λ. Also included are the vacuum increase in wave length, $(n - 1)$, and the refractivity of standard air in the form $(n - 1) \times 1000$.

Authors' Summary

11[W]. Guy H. Orcutt, Martin Greenberger, John Korbel & Alice M. Rivlin, *Microanalysis of Socioeconomic Systems: A Simulation Study*, Harper & Brothers, New York, 1961, xviii + 425 p., 21 cm. Price $8.00.

In this book the authors discuss an experimental calculation carried out on a

high-speed calculator for the purpose of predicting the population trend during the period 1950 to 1960. The calculation is based on an elaborate model which is designed to simulate the demographic characteristics of the population by means of a large number of typical household units (approximately 5000). Each household unit represents a segment of the population, such as the married white female members between the ages of 20 and 25. The calculation proceeds in short time increments (months) until the final state is reached. The distribution of the population is computed at each time interval, taking into consideration the probabilities for such occurrences as births, deaths, divorces, etc. This process may be compared to the use of the Monte Carlo method for the solution of the diffusion equations in physics.

The reviewer believes that the authors would have better served the interests of future research in this field if they had devoted their discussion to a factual description of the results attained and difficulties encountered in carrying out this interesting but rather restricted experiment. However, as indicated by the somewhat pretentious title, this is not the primary emphasis of the book. The authors appear to stress the potential application of their techniques in the simulation of the total social-economic structure of the United States; and the book is promoted as a "pioneer work with a new approach to the scientific study and analysis of social systems, employing the major tools of modern research." The enthusiasm of the authors for their methods would have been more easily understandable if their calculations would have accurately predicted what the population distribution will be in 1970, rather than what it was in 1960.

H. P.

12[W, X]. MELVIN DRESHER, *Games of Strategy: Theory and Applications*, Prentice-Hall, Inc., Englewood Cliffs, New Jersey. 1961, xii + 186 p., 23 cm. Price $9.00.

This small volume on zero-sum two-person games contains essentially the whole story on finite games and a great deal on infinite games. It can be profitably read by anyone with some calculus and the first chapter or so of matrix theory behind him. The author presents an elementary proof of the minimax theorem which also yields a good computational procedure for solving finite games. The properties of optimal strategies are then discussed in an exhaustive and illuminating manner, and various methods of solving games are described. The subject of infinite games, filling one-half the book, is treated next, and the topics covered include games with convex payoff functions, games of timing, and games with separable payoff functions. Numerous examples of such games, described in military terms, are given and their solutions discussed thoroughly. The author's style is pleasant, and the printing and layout of the book are attractive.

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13[X, Z]. L. LAYTON, H. SMITH & L. CHATFIELD, *Proceedings of Executive Seminar on the Use of High-Speed Calculators for the Solution of Naval Problems*, Applied

Mathematics Laboratory, David Taylor Model Basin, Washington 7, D. C., DTMB Report 1519, May 1961, iv + 355 p., 27 cm.

This book consists of twenty unclassified papers presented at a seminar at the Applied Mathematics Laboratory, David Taylor Model Basin, Carderock, Maryland, during 7–9 September 1960. Six additional papers classified "Secret" and one classified "Confidential" are not included in this volume.

The papers are oriented toward the use of high-speed computers in the solution of Naval Problems, with emphasis on applications drawn primarily from the Bureau of Ships activities. The general areas covered are: (1) engineering research, (2) management data analysis, (3) large-scale data processing, (4) operations research, and (5) tactical and strategic planning.

The text is double-spaced and easy to read; however, the quality of reproduction of the photographs leaves much to be desired.

There is not enough space to review each paper separately, so that the following statements may do some injustice to individual papers. Several authors report on their own practical experience, and do not give a perspective to the subject discussed. However, there are many excellent papers, especially "Computer Technology Outside the USA" by Dr. S. N. Alexander, "Nuclear Reactor Design Calculations" by Joanna Wood Schot, "Mathematical Calculation of Shiplines" by Dr. F. Theilheimer, "The Solution of Naval Problems on High-Speed Calculators" by Dr. H. Polachek, and several others. The paper, "On Teaching of Mathematics," by Dr. Francis D. Murnaghan, should be read by every mathematics teacher. All in all, the book offers valuable reading for both the beginner and the experienced computer specialist.

It is unfortunate that the remarks of the keynote speaker, Professor Howard H. Aiken were not recorded for this volume, since he is recognized as the father of modern computers.

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14[Z]. WILHELM KÄMMERER, *Ziffernrechenautomaten*, Akademie-Verlag, Berlin, 1960, viii + 303 p., 24 cm. Price DM 29.

This well-written book is based on a course of lectures given by the author at the Friedrich-Schiller University, in Jena. It discusses computer components, their organization into the various organs of a computer, the logical organization of computers, and the fundamentals of programming.

The first chapter discusses the binary number system and Boolean algebra. The second chapter is concerned with the nature of arithmetic operations and methods for realizing them by automatic devices.

Chapter three deals with the structure of an automatic computer and the requirements that must be imposed on it. It illustrates these requirements and methods by which they have been satisfied, by reference to various computers.

Chapter four discusses in some detail various well-known computer components and methods for organizing them into computer organs. The newer components are not treated.

The final chapter, Chapter five, is devoted to the principles of programming. Several illustrative problems are coded for an imaginary single-address machine. The problems of using a library of subroutines are discussed, as are the notions of relative addresses and floating addresses

The bibliography given at the end of the book is not an extensive one. The oldest references in it are dated 1951. This is somewhat unfortunate, for the reader cannot gain any impression therefrom of the historical development of the subject. The omission of any reference to the fundamental work of von Neumann on computers is, to the reviewer, a great oversight.

A. H. T.

15[Z]. HERBERT D. LEEDS & GERALD M. WEINBERG, *Computer Programming Fundamentals*, McGraw-Hill Book Company Inc., New York, 1961, ix + 368 p., 23 cm. Price $8.50.

Nominally an introductory textbook on digital computing techniques and applications, this book presents a readable account of the basic principles of programming and coding for a specific machine, namely, the IBM 7090 computer. No mathematical knowledge beyond elementary algebra is required. The first section delineates the fundamental characteristics and special capabilities of a computer and then highlights the preparatory steps required to obtain a machine solution. The longer second section is devoted to an exposition of flow-diagramming and coding for the IBM 7090 computer.

In view of the fact that the book is addressed to "students in business administration, economics, and other nontechnical fields as well as the physical sciences and mathematics courses", the authors are disappointingly vague on the subject of programming techniques and procedures for the solution of large-scale data processing problems. Such significant developments as business compilers (COBOL, IBM Commercial Translator, etc.), sort generators, and report generators are not even mentioned. The value of the book as a general text on computer fundamentals is further lessened by the omission of references and supplemental readings. Consequently, the reviewer believes that this volume will be primarily suitable as a general IBM 7090 programming manual for nontechnical readers. It is written in a lively, lucid style that can be easily comprehended by the layman.

MILTON SIEGEL

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16[Z]. W. W. PETERSON, *Error-Correcting Codes*, The Technology Press and John Wiley & Sons, Inc., New York, 1961, x + 285 p., 24 cm. Price $7.75.

The journal literature on algebraic coding theory has become so extensive lately that a book has been needed to give perspective and order to the field. This excellent book not only fills this need but also improves greatly on the presentation in many of the journal articles. In conjunction with the literature on probabilistic schemes of coding and decoding, Peterson's book gives an essentially complete

picture of coding theory as currently known. Most of the material does not appear elsewhere in book form, and a considerable amount is original.

The style of writing is remarkably successful in developing insight and intuition without appreciably sacrificing rigor or conciseness. The book is almost self-contained and includes a development of the required algebraic concepts and theorems. A discussion of ways to implement algebraic operations, particularly on polynomials and Galois field elements by shift register circuits, should help the engineer to understand and use modern algebra, both in coding theory and elsewhere.

The first part of the book discusses linear codes, which are group codes or parity-check codes generalized to include non-binary alphabets. This includes a general treatment, some theoretical bounds on error-correcting ability, and a discussion of several specific classes of linear codes. Next, after some mathematical development, the theory and implementation of cyclic codes is discussed from several different viewpoints.

Bose-Chaudhuri codes, which are the most important of the known algebraic codes, are elegantly treated after cyclic codes. A simple derivation of their error-correcting capabilities is given, and two decoding techniques are presented. The remainder of the book treats burst-error correction, other approaches to decoding, recurrent codes, and the checking of arithmetic operations.

The appendices include a table of irreducible polynomials over the field of two elements. They are arranged in order as minimum polynomials of the elements of Galois field, and this makes it possible to find generator polynomials for Bose-Chaudhuri codes almost by inspection.

The book is highly recommended to engineers and mathematicians interested in coding, information theory, communication, and computers.

R. G. GALLAGER

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17[Z]. A. UNGAR, Proceedings Editor, *Proceedings of the 1959 Computer Applications Symposium*, sponsored by the Armour Research Foundation of Illinois Institute of Technology, Chicago, 1960, x + 155 p., 23 cm. Price $3.00.

Like most symposia, this little book contains quite a variety of papers, some of them excellent, and some of them only mediocre. The paper entitled "Fortran Experience and Remote Operation by Non-computer Specialists" in conjunction with its subsequent panel discussion is, in the opinion of this reviewer, alone well worth the price of the book. This is by no means the only interesting paper. Each paper is followed by the type of bantering discussion that usually takes place in a meeting of a group of specialists.

The papers include: "Shareholder Record-Handling with the Aid of Character-Recognition Equipment," "Around the World in Eighty Columns," "Cost Reduction Through Integrated Data-Processing," "Some Aspects of Computer Technology in the USSR," "Experience and Plans for Marketing-Research Operations," "A Modern Approach to Inventory Control Utilizing a Large-Scale EDPM," "Current Developments in Common-Language Programming for Business Data Systems," "Linear Programming on the Bendix G-15 Computer," "The Design

and Use of the APT Language for Automatic Programming of Numerically Controlled Machine Tools," "A Quasi-Simplex Method for Designing Suboptimum Packages of Electronic Building Blocks," "The International Algebraic Language and the Future of Programming," "Training for Engineers and Scientific Applications via Compilers, Interpreters, and Assemblers," "Scientific Design Procedure Utilizing a Small Computer," and "FORTRAN Experience and Remote Operation by Non-Computer Specialists."

This inexpensive book is recommended for inclusion in the libraries of computation laboratories and of individual programmers.

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TABLE ERRATA

307.—E. P. Adams & R. L. Hippisley, *Smithsonian Mathematical Formulae and Tables of Elliptic Functions*, second and third reprints, The Smithsonian Institution, Washington D. C., 1947 and 1957. See also *MTAC*, v. 12, 1958, p. 262 and earlier references cited there.

The following corrections should be made in section 6 '60, on p. 126:

Formula 4: In the series for $e^{\sin x}$, *for* $+\frac{3x^6}{6!}$, *read* $-\frac{3x^6}{6!}$

Formula 8: In the series for $e^{\tan^{-1}x}$, *for* $+\frac{7x^4}{24}$ *read* $-\frac{7x^4}{24}$.

The corrected series agree with those appearing in B. O. Peirce, *A Short Table of Integrals*, third revised edition, Ginn & Co. Boston, 1929, p. 92–93.

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Editorial Note: For further information regarding the expansion of $e^{\tan^{-1} x}$, see Richard Kelisky, "The numbers generated by exp (arctan x)," *Duke Math. Journal*, v. 26, 1959, p. 569–581, wherein the coefficients of the first twelve powers of x in the expansion are given.

CORRIGENDA

M. Ascher, "Explicit solutions of the one-dimensional heat equation for a composite wall," *Math. Comp.*, v. 14, 1960, p. 346–353.

The last sentence of the derivation of the estimate of error (on page 350) yields the result $O(\Delta x)$ rather than $O(\Delta x^2)$. Nevertheless, the error is $O(\Delta x^2)$, as shown by the numerical example and by abstract No. 576-136, *Amer. Math. Soc. Notices*, v. 7, no. 7, December 1960, p. 944.

Martin Greenberger, "An *a priori* determination of serial correlation in computer generated random numbers," *Math. Comp.*, v. 15, 1961, p. 383–389.

On p. 387, the first two sentences in the paragraph immediately following equation (19) should follow instead equation (11) on p. 385. The reference to equation (1) in the first of these sentences should be changed instead to equation (10).

Sidney Kravitz, "Divisors of Mersenne Numbers $10{,}000 < p < 15{,}000$," *Math. Comp.*, v. 15, 1961, p. 292.

In the 7th column of the table, 8th line down, the number 12,063 should read 13,063.

A.
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CLASSIFICATION OF REVIEWS

A. Arithmetical Tables, Mathematical Constants

B. Powers

C. Logarithms

D. Circular Functions

E. Hyperbolic and Exponential Functions

F. Theory of Numbers

G. Higher Algebra

H. Numerical Solution of Equations

I. Finite Differences, Interpolation

J. Summation of Series

K. Statistics

L. Higher Mathematical Functions

M. Integrals

N. Interest and Investment

O. Actuarial Science

P. Engineering

Q. Astronomy

R. Geodesy

S. Physics, Geophysics, Crystallography

T. Chemistry

U. Navigation

V. Aerodynamics, Hydrodynamics, Ballistics

W. Economics and Social Sciences

X. Numerical Analysis and Applied Mathematics

Z. Calculating Machines and Mechanical Computation

Mathematics of Computation

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